Extending Drawings of Graphs to Arrangements of Pseudolines

Alan Arroyo
IST Austria, Klosterneuburg, Austria
https://alanarroyo.github.io/
alanmarcelo.arroyoguevara@ist.ac.at

Julien Bensmail
Université Côte d’Azur, CNRS, Inria, I3S, Sophia-Antipolis, France
julien.bensmail.phd@gmail.com

R. Bruce Richter
Department of Combinatorics and Optimization, University of Waterloo, Canada
brichter@uwaterloo.ca

Abstract
In the recent study of crossing numbers, drawings of graphs that can be extended to an arrangement of pseudolines (pseudolinear drawings) have played an important role as they are a natural combinatorial extension of rectilinear (or straight-line) drawings. A characterization of the pseudolinear drawings of $K_n$ was found recently. We extend this characterization to all graphs, by describing the set of minimal forbidden subdrawings for pseudolinear drawings. Our characterization also leads to a polynomial-time algorithm to recognize pseudolinear drawings and construct the pseudolines when it is possible.

1 Introduction
Since 2004, geometric methods have been used to make impressive progress for determining the crossing number of (certain classes of drawings of) the complete graph $K_n$. In particular, drawings that extend to straight lines, or, more generally, arrangements of pseudolines, have been central to this work, spurring interest in such drawings for arbitrary graphs, not just complete graphs [2, 5, 6, 7, 12].

In particular, for pseudolinear drawings, it is now known that, for $n \geq 10$, a pseudolinear drawing of $K_n$ has more than

$$H(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

crossings [1, 13]. The number $H(n)$ is conjectured by Harary and Hill to be the smallest number of crossings over all topological drawings of $K_n$; that is, the crossing number $\text{cr}(K_n)$ is conjectured to be $H(n)$.
A pseudoline is the image $\ell$ of a continuous injection from the real numbers $\mathbb{R}$ to the plane $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \ell$ is not connected. An arrangement of pseudolines is a set $\Sigma$ of pseudolines such that, if $\ell, \ell'$ are distinct elements of $\Sigma$, then $|\ell \cap \ell'| = 1$ and the intersection is a crossing point. More on pseudolines and their importance for studying geometric drawings of graphs can be found in [10, 11].

A drawing $D$ of a graph $G$ is pseudolinear if there is an arrangement of pseudolines consisting of a different pseudoline $\ell_e$ for each edge $e$ of $G$ and such that $D[e] \subseteq \ell_e$.

In the study of crossing numbers, restricting the drawing to either straight lines or pseudolines yields the rectilinear crossing number $\overline{cr}(K_n)$ or the pseudolinear crossing number $\tilde{cr}(K_n)$, respectively. Clearly $\overline{cr}(K_n) \geq \tilde{cr}(K_n)$ and the geometric methods prove that $\tilde{cr}(K_n) > H(n)$, for $n \geq 10$.

A good drawing is one where no edge self-intersects and any two edges share at most one point – either a crossing or a common end point – and no three edges share a common crossing. One somewhat surprising result is from Aichholzer et al.: a good drawing of $K_n$ in the plane is homeomorphic to a pseudolinear drawing if and only if it does not contain a non-planar drawing of $K_4$ whose crossing is incident with the unbounded face of the $K_4$ [2]. There are equivalent characterizations in [5, 6]. These conditions can be shown to be equivalent to not containing the $B$-configuration depicted as the third drawing of the first row of Figure 1.

Twenty-five years earlier, Thomassen proved a similar theorem for drawings in which each edge is crossed at most once [16]. The $B$- and $W$-configurations are shown as the third and fourth drawings in the first row of Figure 1. Thomassen’s theorem is: if $D$ is a planar drawing of a graph $G$ in which each edge is crossed at most once, then $D$ is homeomorphic to a rectilinear drawing of $G$ if and only if $D$ contains no $B$- or $W$-configuration.

Thomassen presented in [16] the clouds (first column in Figure 1) as an infinite family of drawings that are minimally non-pseudolinear.

Shortly after Thomassen’s paper, Bienstock and Dean proved that if $\text{cr}(G) \leq 3$, then $\overline{cr}(G) = \text{cr}(G)$ [8]. They also exhibited examples based on overlapping $W$-configurations to show the result fails for $\text{cr}(G) = 4$; such graphs can have arbitrarily large rectilinear crossing number.

Despite the existence of infinitely many obstructions to pseudolinearity, we characterize them all.

**Theorem 1.** A good drawing of a graph $G$ is pseudolinear if and only if it does not contain one of the infinitely many obstructions shown in Figure 1.
The drawings in Figure 1 are obtained from the clouds (first column) by replacing at most two crossings by vertices. The formal statement of Theorem 1 is Theorem 15 in Section 6; also a more general version of this statement, Theorem 2, is discussed below. That there is a result such as ours is somewhat surprising, because stretching an arrangement of pseudolines to a rectilinear drawing has been shown by Mnëv [14, 15] to be \( \exists R - \text{hard} \). In particular, recognizing a drawing as being homeomorphic to a rectilinear drawing is \( \text{NP}-\text{hard} \).

The natural setting for our characterization is strings embedded in the plane. An arc \( \sigma \) is the image \( f([0, 1]) \) of the compact interval \([0, 1]\) under a continuous map \( f : [0, 1] \rightarrow \mathbb{R}^2 \). Let \( S(\sigma) = \{ p \in \sigma : |f^{-1}(p)| \geq 2 \} \) be the set of self-intersections of \( \sigma \). A string is an arc \( \sigma \) for which \( S(\sigma) \) is finite. If \( S(\sigma) = \emptyset \), then \( \sigma \) is simple.

An intersection point between of two strings \( \sigma \) and \( \sigma' \) is ordinary if it is either an endpoint of \( \sigma \) or \( \sigma' \), or is a crossing (a crossing is a non-tangential intersection point in \( \sigma \cap \sigma' \) that is not an end of \( \sigma \) or \( \sigma' \)). A set \( \Sigma \) of strings is ordinary if \( \Sigma \) is finite and any two strings in \( \Sigma \) have only finitely many intersections, all of which are ordinary. All the sets of strings considered in this paper are ordinary.

If \( \Sigma \) is an ordinary set of strings, then its planarization \( G(\Sigma) \) is the plane graph obtained from \( \Sigma \) by inserting vertices at each crossing between strings and also at the endpoints of every string in \( \Sigma \). To keep track of the information given by the strings, we will always assume that each string \( \sigma \) has a different color and that each edge in \( G(\Sigma) \) inherits the color of the string including it.

If \( \Sigma \) is an ordinary set of strings, then, for a cycle \( C \) in \( G(\Sigma) \) (which is a simple closed curve in \( \mathbb{R}^2 \)) and a vertex \( v \in V(C) \), \( v \) is a rainbow for \( C \) if all the edges incident with \( v \) and drawn in the closed disk bounded by \( C \) (including the two edges of \( C \) at \( v \)) have different colours. The reader can verify that, for each drawing in Figure 1, if we let \( \Sigma \) be the edges \( \Sigma \) has a different color and that each edge in \( G(\Sigma) \) inherits the color of the string including it.

A set of strings with an obstruction is not extendible

Let us start by showing the easy direction of Theorem 2:

\( \text{Lemma 3.} \) If the underlying graph \( G(\Sigma) \) of a set \( \Sigma \) of strings has an obstruction, then \( \Sigma \) is not pseudolinear.
Suppose that $C$ is a cycle of $G(\Sigma)$ for some set of strings $\Sigma$. We define $\delta(C)$ as the set of vertices of $C$ for which their two incident edges in $C$ have different colours. In a set $\Sigma$ of simple strings where no two intersect twice, $|\delta(C)| \geq 3$ for every cycle $C$ of $G(\Sigma)$.

**Lemma 4.** Let $\Sigma$ be a set of simple strings where every pair intersect at most once. Suppose that $C$ is an obstruction with $|\delta(C)|$ as small as possible. Let $S = x_0, x_1, \ldots, x_\ell$ be a path of $G(\Sigma)$ representing a subsegment of some string $\sigma \in \Sigma$ such that $x_0x_1 \in E(C)$, $x_1 \in \delta(C)$ and $x_1$ is not a rainbow of $C$. Then $V(C) \cap V(S) = \{x_0, x_1\}$.

**Proof.** By way of contradiction, suppose that there is a vertex $x_r \in V(C) \cap V(S)$ with $r \geq 3$. Assume that $r \geq 3$ is as small as possible. Let $P$ be the subpath of $S$ connecting $x_1$ to $x_r$. Since $x_0x_1 \in E(C)$ and $x_1 \in \delta(C)$ and $P \subseteq \sigma$, $x_1x_2 \notin E(C)$. Because $x_1$ is not a rainbow for $C$ and no two strings tangentially intersect at $x_1$, the edge $x_1x_2$ is drawn in the closed disk bounded by $C$. By choice of $r$, $P$ is an arc connecting $x_1$ to $x_r$ in the interior of $C$.

Let $C_1$ and $C_2$ be the cycles obtained from the union of $P$ and one of the two $xy$-subpaths in $C$. We may assume that $x_0x_1 \in E(C_1)$. Let $\rho(C)$ be either $\delta(C)$ or the set of rainbows in $C$. For $i = 1, 2$, let $Q_i = V(C_i) \setminus V(P)$. Then $\rho(C) \cap Q_i = \rho(C_i) \cap Q_i$. We see that $\rho(C_1) \cap Q_1 \subseteq \{x_r\}$ and $\rho(C_2) \cap Q_2 \subseteq \{x_1, x_r\}$.

For $\rho = \delta$, $|\delta(C_2)| \geq 3$, so $|\delta(C) \cap Q_2| \geq 1$. Since $x_1 \notin \delta(C_1)$, $|\delta(C_1)| \leq |\delta(C_1) \cap Q_2| + |\{x_r\} | \leq |\delta(C)| - 2 + |\{x_r\} | < |\delta(C)|$. Likewise, $|\delta(C) \cap Q_1| \geq 2$ and $x_1 \in \delta(C) \cap \delta(C_2)$. Therefore, $|\delta(C_2)| \leq |\delta(C)| - 2 + |\{x_r\} | < |\delta(C)|$. Thus, neither $C_1$ nor $C_2$ is an obstruction.

Now taking $\rho$ to be the set of rainbows, the preceding paragraph shows $|\rho(C_1)| \geq 3$ and $|\rho(C_2)| \geq 3$. Therefore, $|\rho(C) \cap Q_1| = |\rho(C_1) \cap Q_1| \geq 2$ and $|\rho(C) \cap Q_2| = |\rho(C_2) \cap Q_2| \geq 1$. Thus, $|\rho(C)| \geq 3$, a contradiction.

**Proof of Lemma 3.** By way of contradiction, suppose that $\Sigma$ is pseudolinear and that $G(\Sigma)$ has an obstruction $C$. Consider an extension of $\Sigma$ to an arrangement of pseudolines, and then cut off the two infinite ends of each pseudoline to obtain a set of strings $\Sigma'$ extending $\Sigma$, and in which every pair of strings in $\Sigma'$ cross once. In $G(\Sigma')$, there is a cycle $C'$ that represents the same simple closed curve as $C$. Because $C'$ is obtained from subdividing some edges of $C$ and the colours of a subdivided edge are the same, $C'$ has fewer than three rainbows. Therefore, we may assume that $\Sigma = \Sigma'$ and $C = C'$. Now, the ends of every string in $\Sigma$ are degree-1 vertices in the outer face of $G(\Sigma)$.

As every string in $\Sigma$ is simple and no two strings intersect more than once, $|\delta(C)| \geq 3$. We will assume that $C$ is chosen to minimize $|\delta(C)|$.

Since $C$ is an obstruction, there exists $x_1 \in \delta(C)$ such that $x_1$ is not a rainbow in $C$. Consider a neighbour $x_0$ of $x_1$ in $C$. Let $S = x_0, x_1, \ldots, x_\ell$ be the path obtained by traversing the string $\sigma$ extending $x_0x_1$, such that $x_\ell$ is an end of $\sigma$. By Observation 4, $V(S) \cap V(C) = \{x_0, x_1\}$, and because $x_1$ is in the outer face of $C$, the segment of $\sigma$ from $x_1$ to $x_\ell$ has its relative interior in the outer face of $C$.

However, since $x_1$ is not a rainbow, there exists a string $\sigma' \in \Sigma$ including two edges at $x_1$ drawn in the disk bounded by $C$. Thus, $\sigma$ and $\sigma'$ tangentially intersect at $x_1$, a contradiction.

**3 The key lemma**

In this section we present the key lemma used in the proof of Theorem 2.

A plane graph $G$ is path-partitioned if for $m \geq 1$, there exists a colouring $\chi : E(G) \rightarrow \{1, \ldots, m\}$ such that for each $i \in \{1, \ldots, m\}$, the edges in $\chi^{-1}(i)$ induce a path $P_i \subseteq G$ where any two distinct paths $P_i$ and $P_j$ do not tangentially intersect. Indeed, every underlying
planar graph \( G(\Sigma) \) of a set of simple strings \( \Sigma \) is path-partitioned. Moreover, every path-partitioned plane graph can be obtained by subdividing a planarization of an ordinary set of simple strings. To extend the previously introduced notation we refer to each \( P_i \) as a string. The concepts of rainbow and obstruction naturally extend to the context of path-partitioned plane graphs.

Suppose that \( G \) is a path-partitioned plane graph. Given \( v \in V(G) \), a near-obstruction at \( v \) is a cycle \( C \) with at most three rainbows and such that \( v \) is a rainbow of \( C \). Understanding how near-obstructions behave is the key ingredient needed in the proof of Theorem 2:

**Lemma 5.** Let \( G \) be a path-partitioned plane graph and let \( v \in V(G) \). Suppose that \( C_1 \) and \( C_2 \) are two near-obstructions at \( v \) such that the union of the closed disks bounded by \( C_1 \) and \( C_2 \) contains a small open ball centered at \( v \). Suppose that one of the following two holds:

1. no obstruction of \( G \) contains \( v \); or
2. the two edges of \( C_1 \) incident with \( v \) are the same as the two edges of \( C_2 \) incident with \( v \).

Then \( G \) has an obstruction not including \( v \).

Given a plane graph \( G \), a cycle \( C \subseteq G \) and a vertex \( v \in V(C) \), the edges at \( v \) inside \( C \) are the edges of \( G \) incident with \( v \) drawn in the disk bounded by \( C \).

**Useful Fact.** Let \( G \) be planar path-partitioned graph. Suppose that for two cycles \( C \) and \( C' \), \( v \in V(C) \cap V(C') \) is a vertex such that the edges at \( v \) inside \( C' \) are also edges at \( v \) inside \( C \). If \( v \) is a rainbow for \( C \), then \( v \) is a rainbow for \( C' \).

**Proof of Lemma 5.** By way of contradiction, suppose that \( G \) has no obstruction not including \( v \). The “small ball” hypothesis implies that \( v \) is not in the outer face of the subgraph \( C_1 \cup C_2 \).

We claim that \( |V(C_1) \cap V(C_2)| \geq 2 \). Suppose not. Then \( C_1 \) and \( C_2 \) are edge-disjoint and \( V(C_1) \cap V(C_2) = \{v\} \). For \( i = 1, 2 \), let \( e_i \) and \( f_i \) be the edges of \( C_i \) at \( v \) and let \( \Delta_i \) be the closed disk bounded by \( C_i \). From the “small ball” hypothesis it follows that (i) \( \Delta_1 \) contains the edges \( e_2 \) and \( f_2 \); and (ii) the points near \( v \) in the exterior of \( \Delta_2 \) are contained in \( \Delta_1 \). These two properties imply that the path \( C_2 - \{e_2, f_2\} \) intersects \( C_1 \) at least twice, and hence, \( |V(C_1) \cap V(C_2)| \geq 2 \).

From the last paragraph we know that \( C_1 \cup C_2 \) is 2-connected, and hence the outer face of \( C_1 \cup C_2 \) is bounded by a cycle \( C_{out} \). We will assume that

\[ (*) \quad \text{the cycles } C_1 \text{ and } C_2 \text{ satisfying the hypothesis of Lemma 5 are chosen so that the number of vertices of } G \text{ in the disk bounded by } C_{out} \text{ is minimal.} \]

The Useful Fact applied to \( C = C_{out} \) and to each \( C' \in \{C_1, C_2\} \), shows that every vertex that is a rainbow in \( C_{out} \) is also a rainbow in each of the cycles in \( \{C_1, C_2\} \) containing it. We can assume that \( C_{out} \) is not an obstruction or else we are done. We may relabel \( C_1 \) and \( C_2 \) so that two of the rainbows of \( C_{out} \), say \( p \) and \( q \), are also rainbows in \( C_1 \). Neither \( p \) nor \( q \) is \( v \) because \( v \notin V(C_{out}) \). Because \( C_1 \) is a near-obstruction, \( p, q \) and \( v \) are the only rainbows of \( C_1 \).

Since \( v \notin V(C_{out}) \), by following \( C_1 \) in the two directions starting at \( v \), we find a path \( P_v \subseteq C_1 \) containing \( v \) in which only the ends \( u \) and \( w \) of \( P_v \) are in \( C_{out} \) (note that \( u \neq v \) because \( \{p, q\} \subseteq V(C_1) \cap V(C_{out}) \)). As \( v \) is in the interior face of \( C_{out} \), \( P_v \) is also in the interior of \( C_{out} \). Let \( Q_{out}^1, Q_{out}^2 \) be the \( uw \)-paths of \( C_{out} \). One of the two closed disks bounded by \( P_v \cup Q_{out}^1 \) and \( P_v \cup Q_{out}^2 \) contains \( C_1 \). By symmetry, we may assume that \( C_1 \) is contained in the first disk. Since \( C_{out} \subseteq C_1 \cup C_2 \), this implies that \( Q_{out}^2 \) is a subpath of \( C_2 \).
Our desired contradiction will be to find three rainbows in $C_2$ distinct from $v$. We find the first: let $C_1 = (P_u)$ be the $uv$-path in $C_1$ distinct from $P_u$. The disk bounded by $(C_1 - (P_u)) \cup Q^2_{\text{out}}$ contains the one bounded by $C_1$. The Useful Fact applied to $C = (C_1 - (P_u)) \cup Q^2_{\text{out}}$ and $C' = C_1$ implies that each vertex in $C_1 - (P_u)$ that is rainbow in $(C_1 - (P_u)) \cup Q^2_{\text{out}}$ is also rainbow in $C_1$. Since $C_1$ has at most two rainbows in $C_1 - (P_u)$, namely $p$ and $q$, $(C_1 - (P_u)) \cup Q^2_{\text{out}}$ has a third rainbow $r_1$ in the interior of $Q^2_{\text{out}}$ (else $(C_1 - (P_u)) \cup Q^2_{\text{out}}$ is an obstruction and we are done). Note that $r_1$ is also a rainbow for $C_2$.

To find another rainbow in $C_2$, consider the edge $e_u$ of $C_2$ incident to $u$ and not in $Q^2_{\text{out}}$. We claim that either $u$ is a rainbow in $C_2$ or that $e_u$ is not included in the closed disk bounded by $P_u \cup Q^2_{\text{out}}$. Seeking a contradiction, suppose that $u$ is not a rainbow of $C_2$ and that $e_u$ is included in the disk. Then we can find two edges in the rotation at $u$, included in the disk bounded by $P_u \cup Q^2_{\text{out}}$, that belong to the same string $\sigma$. The vertex $v$ is a rainbow in $C_1$, as else, we would find a string $\sigma'$ with two edges inside $Q^1_{\text{out}} \cup P_v$, showing that $\sigma$ and $\sigma'$ tangentially intersect at $u$. If $p$ and $q$ are the only rainbows of $C_1$ in $C_{\text{out}}$, $u$ is one of $p$ and $q$. Therefore $u$ is a rainbow in $C_{\text{out}}$, and hence, a rainbow in $C_2$, a contradiction.

If $u$ is a rainbow in $C_2$, then this is the desired second one. Otherwise, $e_u$ is not in the closed disk bounded by $P_u \cup Q^2_{\text{out}}$. Let $P_u \subseteq C_2$ be the path starting at $u$, continuing on $e_u$ and ending on the first vertex $u'$ in $P_u$ that we encounter. Let $C_u$ be the cycle consisting of $P_u$ and the $uu'$-subpath $uP_vu'$ of $P_v$.

> Claim 6. If $P_u$ does not have a rainbow of $C_u$ in its interior, then either $C_u$ is an obstruction not containing $v$ or:

(a) $C_u$ and $C_2$ are near-obstructions at $v$ satisfying the same conditions as $C_1$ and $C_2$ in Lemma 5; and

(b) the closed disk bounded by the outer cycle of $C_u \cup C_2$ contains fewer vertices than the disk bounded by $C_{\text{out}}$.

Proof. Suppose that all the rainbows of $C_u$ are located in $uP_vu'$. If $z$ is a rainbow of $C_u$, then $z \in \{u, v, u'\}$, as otherwise $z$ is a rainbow of $C_1$ distinct from $p, q$ and $v$, a contradiction. Thus, if $v \notin V(C_u)$, then $C_u$ is the desired obstruction. We may assume that $v \in V(C_u)$.

If $u' = w$, then $C_2 = P_u \cup Q^2_{\text{out}}$, violating the assumption that $v \in V(C_2)$. Thus $u' \neq w$. If $u' = v$, then the rainbows of $C_u$ are included in $\{u, u'\}$, and hence $C_u$ is an obstruction. However, the existence of $C_u$ shows that both alternatives (1) and (2) in Lemma 5 fail: condition (1) fails because $C_u$ contains $v$ and (2) fails because the edge of $P_u$ incident with $v$ is in $E(C_2) \cup E(C_1)$. Thus $u' \neq v$.

The previous two paragraphs show that $C_u$ is a near-obstruction at $v$ with rainbows $u, v$ and $u'$. Since the interior of $C_u$ near $v$ is the same as the interior of $C_1$ near $v$, the pair $(C_u, C_2)$ satisfies the “small ball” hypothesis. Thus, (a) holds.

Let $C_{\text{out}}$ be the outer cycle of $C_u \cup C_2$. From the fact that $C_u \cup C_2 \subseteq C_1 \cup C_2$ it follows that the disk bounded by $C_{\text{out}}$ includes the disk bounded by $C_{\text{out}}$.

Since $p, q \in V(C_{\text{out}})$, $p$ and $q$ are in the disk bounded by $C_{\text{out}}$. If both $p$ and $q$ are in $C_2$, then $p, q$ and $r_1$ are rainbows in $C_2$, and also distinct from $v$, contradicting that $C_2$ is a near-obstruction for $v$. If, say $p \notin V(C_2)$, then $p$ is not in the disk bounded by $C_{\text{out}}$, which implies (b).

From Claim 6(b) and assumption (*) either $C_u$ is the desired obstruction or $P_u$ contains a rainbow $r_2$ of $C_2$ in its interior. We assume the latter as else we are done.

In the same way, the last rainbow $r_3$ comes by considering the edge of $C_2 - Q^2_{\text{out}}$ incident with $w$. It follows that $v, r_1, r_2$ and $r_3$ are four different rainbows in $C_2$, contradicting the fact that $C_2$ is a near-obstruction.
4 Proof of Theorem 2

In this section we prove that a set of strings with no obstructions can be extended to an arrangement of pseudolines.

Proof of Theorem 2. It was shown in Observation 3 that the existence of obstructions implies non-extendibility. For the converse, suppose that $\Sigma$ is a set of strings for which $G(\Sigma)$ has no obstructions.

We start by reducing to the case where the point set $\bigcup \Sigma$ is connected: iteratively add a new string in a face of $\bigcup \Sigma$ connecting two connected components of $\bigcup \Sigma$. No obstruction is introduced at each step (obstructions are cycles), and, eventually, the obtained set $\bigcup \Sigma$ is connected. An extension of the new set of strings contains an extension for the original set, thus we may assume that $\bigcup \Sigma$ is connected.

Our proof is algorithmic, and consists of repeatedly applying one of the three steps described below.

- **Disentangling Step.** If a string $\sigma \in \Sigma$ has an end $a$ with degree at least 2 in $G(\Sigma)$, then we slightly extend the $a$-end of $\sigma$ into one of the faces incident with $a$.

- **Face-Escaping Step.** If a string $\sigma \in \Sigma$ has an end $a$ with degree 1 in $G(\Sigma)$, and is incident with an interior face, then we extend the $a$-end of $\sigma$ until it intersects some point in the boundary of this face.

- **Exterior-Meeting Step.** Assuming that all the strings in $\Sigma$ have their two ends in the outer face and these ends have degree 1 in $G(\Sigma)$, we extend the ends of two disjoint strings so that they meet in the outer face.

Each of these three steps either increases the number of pairs of strings that intersect, or increase the number crossings (recall that a crossing between $\sigma$ and $\sigma'$ is a non-tangential intersection point in $\sigma \cap \sigma'$ that is not an end of $\sigma$ or $\sigma'$). Moreover, these steps can be performed as long as not all the strings have their ends in the outer face and they are pairwise crossing (in this case we extend their ends to infinity to obtain the desired arrangement of pseudolines). Henceforth, we will show that, if performed correctly, none of these steps introduces an obstruction. The proof for each step can be read independently.

Lemma 7 (Disentangling Step). Suppose that $\sigma \in \Sigma$ has an end $a$ with degree at least 2 in $G(\Sigma)$. Then we can extend the $a$-end of $\sigma$ into one of the faces incident to $a$ without creating an obstruction.

Proof. A pair of different edges $f$ and $f'$ in $G(\Sigma)$ incident with $a$ are twins if they belong to the same string in $\Sigma$. The edge $e \subseteq \sigma$ incident with $a$ has no twin.

The fact that no pair of strings tangentially intersect at $a$ tells us that if $(f_i, f'_i)$ and $(f_2, f'_2)$ are pairs of twins, then $f_1, f_2, f'_1, f'_2$ occur in this cyclic order for either the clockwise or counterclockwise rotation at $a$. Thus, we may assume that the counterclockwise rotation at $a$ restricted to the twins and $e$ is $e, f_1, \ldots, f_t, f'_1, \ldots, f'_t$, where $(f_i, f'_i)$ is a twin pair for $i = 1, \ldots, t$.

To avoid tangential intersections, the extension of $\sigma$ at $a$ must be in the angle between $f_t$ and $f'_t$ not containing $e$. Let $e_1, \ldots, e_k$ be the counterclockwise ordered list of non-twin edges at $a$ having an end in this angle (as depicted in Figure 2). We label $e_0 = f_t$ and $e_{k+1} = f'_t$. If there are no twins, then let $e_0 = e_{k+1} = e$.

Let us consider all the possible extensions: for $i \in \{0, \ldots, k\}$, let $\Sigma_i$ be the set of strings obtained from $\Sigma$ by slightly extending the $a$-end of $\sigma$ into the face containing the angle between $e_i$ and $e_{i+1}$. Let $\alpha_i$ be the new edge at $a$ extending $\sigma$ in $\Sigma_i$ (see $\alpha_0$ in Figure 2).
Seeking a contradiction, suppose that, for each \( i \in \{0, \ldots, k\} \), \( G(\Sigma_i) \) contains an obstruction \( C_i \). Since \( \alpha_i \) contains a degree-1 vertex, \( \alpha_i \) is not in \( C_i \). Hence \( C_i \) is a cycle of \( G(\Sigma) \). Thus \( C_i \) is not an obstruction in \( G(\Sigma) \) that becomes one in \( G(\Sigma_i) \). This conversion has a simple explanation: in \( G(\Sigma) \), \( C_i \) has exactly three rainbows, and one of them is \( a \). After \( \alpha_i \) is added, \( a \) is not a rainbow in \( C_i \) (witnessed by the edges \( e \) and \( \alpha_i \) included in the new version of \( \sigma \)).

Recall from Section 3 that a near-obstruction at \( a \) is a cycle with exactly three rainbows, and one of them is \( a \). Each of \( C_0 \), \( C_1 \), \ldots, \( C_k \) is a near-obstruction at \( a \) in \( G(\Sigma) \).

For a cycle \( C \subseteq G \), let \( \Delta(C) \) denote the closed disk bounded by \( C \). Both \( e \) and \( \alpha_0 \) are in \( \Delta(C_0) \). Thus, either \( \Delta(C_0) \supseteq \{e, f_1, f_2, \ldots, f_t, e_1\} \) (blue bidirectional arrow in Figure 2) or \( \Delta(C_0) \supseteq \{f_t, e_1, \ldots, e_k, f'_1, f'_2, \ldots, f'_t, e\} \) (green bidirectional arrow). We rule out the latter situation as the second list contains \( f_t \) and \( f'_t \), and this would imply that \( a \) is not a rainbow for \( C_0 \) in \( G(\Sigma) \).

We just showed that \( \{e, e_0, e_1\} \subseteq \Delta(C_0) \). By symmetry, \( \{e_k, e_{k+1}, e\} \subseteq \Delta(C_k) \). Consider the largest index \( i \in \{0, 1, \ldots, k-1\} \) for which \( \{e, e_0, \ldots, e_{i+1}\} \subseteq \Delta(C_i) \). By the choice of \( i \), and because \( \{e, \alpha_{i+1}\} \subseteq \Delta(C_{i+1}) \), \( \{e, f'_1, \ldots, f'_t, e_{k}, \ldots, e_i\} \subseteq \Delta(C_{i+1}) \). However, by applying Lemma 5 to the pair \( C_i \) and \( C_{i+1} \), we obtain that \( G(\Sigma) \) has an obstruction, a contradiction.

**Lemma 8 (Face-Escaping Step).** Suppose that there is a string \( \sigma \) that has an end \( a \) with degree 1 in \( G(\Sigma) \), and \( a \) is incident to an interior face \( F \). Then there is an extension \( \sigma' \) of \( \sigma \) from its \( a \)-end to a point in the boundary of \( F \) such that the set \( (\Sigma \setminus \{\sigma\}) \cup \{\sigma'\} \) has no obstruction.

**Figure 3** All possible extensions in the Face-Escaping Step.
Proof. Let $W$ be the closed boundary walk $(x_0, e_1, x_1, e_2, \ldots, e_n, x_n)$ of $F$ such that $x_0 = x_n = a$ and $F$ is to the left as we traverse $W$ (see Figure 3 for an illustration with $n = 9$). For $i = 1, \ldots, n$ we let $m_i$ be a point in the relative interior of $e_i$, and let $P$ be the list of non-necessarily distinct points $m_1, x_1, m_2, x_2, \ldots, m_n$, which are the potential ends for all the different extensions. For each $p \in P$, let $\Sigma_p$ be the set of strings obtained from $\Sigma$ by extending the $a$-end of $\sigma$ by adding an arc $\alpha_p$ connecting $a$ to $p$ in $F$ (see Figure 3). We assume that every two distinct arcs $\alpha_p$ and $\alpha_{p'}$ are internally disjoint.

Let $f_p$ be the edge $e_1 \cup \alpha_p$ in $G(\Sigma_p)$; $f_p$ has ends $x_1$ and $p$. Also, let $\sigma^p = \sigma \cup \alpha_p$. Seeking a contradiction, suppose that each $G(\Sigma_p)$ has an obstruction.

\begin{itemize}
\item[$\triangleright$] Claim 9. Let $p \in P$. Then there exists an obstruction $C_p$ in $G(\Sigma_p)$ including $f_p$. Moreover, (1) if $p \in \sigma$, then $C_p$ can be chosen so that all its edges are included in $\sigma^p$; and (2) if $p \notin \sigma$, then every obstruction includes $f_p$.
\end{itemize}

\begin{proof}
First, if $p \in \sigma$, then the string $\sigma^p$ self-intersects at $p$; thus $\sigma^p$ has a simple close curve including $f_p$. In this case let $C_p$ be the cycle in $G(\Sigma_p)$ representing this simple closed curve without rainbows, and thus (1) holds.

Second, assume that $p \notin \sigma$ and let $C_p$ be any obstruction of $G(\Sigma_p)$. For (2), we will show that $f_p \in E(C_p)$.

Seeking a contradiction, suppose that $f_p \notin E(C_p)$.

If $p = m_i$ for $i \in \{1, \ldots, n\}$, since $m_i$ is the only vertex whose rotation in $G(\Sigma)$ differs from its rotation in $G(\Sigma_{m_i})$, $m_i \in V(C_p)$. Consider the cycle $C'$ of $G(\Sigma)$ obtained from $C_p$ by replacing the subpath $(x_{i-1}, m_i, x_i)$ by the edge $x_{i-1}x_i$. For each vertex $v \in V(C')$ the colors of the edges of $G(\Sigma)$ at $v$ included in the disk bounded by $C'$ are the same as in $G(\Sigma_p)$ for the disk bounded by $V(C_p)$. Thus, $C'$ is an obstruction for $G(\Sigma)$, a contradiction.

Suppose now that $p$ is one of $x_1, \ldots, x_{n-1}$. The only vertex in $G(\Sigma)$ whose rotation is different in $G(\Sigma_p)$ is $p$. Therefore, $p$ is a point that is a rainbow for $C_p$ in $G(\Sigma)$, but not a rainbow in $G(\Sigma_p)$, witnessed by two edges included in $\sigma^p$. Since at least one of the two witnessing edges is in $G(\Sigma)$, $p \in \sigma$. This contradicts the assumption that $p \notin \sigma$. Hence $f_p \in E(C_p)$.

Henceforth we assume that, for $p \in P$, $C_p$ is an obstruction in $G(\Sigma_p)$ as in Claim 9.

More can be said about the obstructions in $G(\Sigma_p)$, but for this we need some terminology. If we orient an edge $e$ in a plane graph, then the sides of $e$ are either the points near $e$ that are to the right of $e$, or the points near $e$ to the left of $e$. For any cycle $C$ of $G$ through $e$, exactly one side of $e$ lies inside $C$. This is the side of $e$ covered by $C$. For the next claim and in the rest of the proof we will assume that for $p \in P$, $f_p$ is oriented from $x_1$ to $p$.

\begin{itemize}
\item[$\triangleright$] Claim 10. For $p \in P$ with $p \notin \sigma$, every obstruction in $G(\Sigma_p)$ covers the same side of $f_p$.
\end{itemize}

\begin{proof}
Suppose that for $p \in P$ there are obstructions $C_p$ and $C'_p$ covering both sides of $f_p$. Let $G'$ be the plane graph obtained from $G(\Sigma_p)$ by subdividing $f_p$, and let $v$ be the new degree-2 vertex inside $f_p$.

We consider the edge-colouring $\chi$ induced by the strings in $\Sigma_p$. Let $\chi'$ be a new colouring obtained from $\chi$ by replacing the colour of the edge $vp$ by a new colour not used in $\chi$. It is a routine exercise to verify that (i) $\chi'$ induces a path-partition in $G'$ (defined in Section 3); and (ii) $C_p$ and $C'_p$ are near-obstructions for $v$ with respect to $\chi'$. By applying Lemma 5 to $C_1 = C_p$ and $C_2 = C'_p$, we obtain an obstruction in $G'$ not containing $v$. However, this implies the existence of an obstruction in $G(\Sigma)$, a contradiction.

\end{proof}
Recall that the boundary walk of $F$ is $W = (x_0, e_1, \ldots, e_n, x_n)$, with $x_0 = x_n = a$. Since $x_1$ and $x_{n-1}$ are in $\sigma$, the extreme obstructions $C_{x_1}$ and $C_{x_{n-1}}$ cover the right of $f_{x_1}$ and the left of $f_{x_{n-1}}$, respectively. Thus, there are two consecutive vertices $x_{i-1}$, $x_i$ in $W - a$, such that the interior of $C_{x_{i-1}}$ covers the right of $f_{x_{i-1}}$ and the interior of $C_{x_i}$ covers the left of $f_{x_i}$. Moreover, we may assume that the interior of $C_{m_i}$ includes the left of $f_{m_i}$ (otherwise we reflect our drawing).

The next claim (proved in the full version of this paper [4]) is the last ingredient to obtain a final contradiction.

> Claim 11. Exactly one of the following holds:

(a) $x_{i-1} \in \sigma$, $m_i \notin \sigma$ and $G(\Sigma_{m_i})$ has an obstruction covering the side of $f_{m_i}$ not covered by $C_{m_i}$; or
(b) $x_{i-1} \notin \sigma$ and $G(\Sigma_{x_{i-1}})$ has an obstruction covering the side of $f_{x_{i-1}}$ not covered by $C_{x_{i-1}}$.

Claims 10 and 11 contradict each other. Thus, for some $p \in P$, $G(\Sigma_p)$ has no obstructions. ◀

> Lemma 12 (Exterior-Meeting Step). If all the strings in $\Sigma$ have their ends on the outer face of $G(\Sigma)$ and the ends have degree 1 in $G(\Sigma)$, then we can extend a pair disjoint strings so that they intersect without creating an obstruction.

**Proof.** By following the outer boundary of $\bigcup \Sigma$, we obtain a simple closed curve $O$ containing all the ends of the strings in $\Sigma$, but otherwise disjoint from $\bigcup \Sigma$.

Suppose $\sigma_1$, $\sigma_2$ are two disjoint strings in $\Sigma$. For $i = 1, 2$, let $a_i$, $b_i$ be the ends of $\sigma_i$; since $\sigma_1$ and $\sigma_2$ do not cross, we may assume that these ends occur in the cyclic order $a_1$, $b_1$, $b_2$, $a_2$. We extend the $a_i$-ends of $\sigma_1$ and $\sigma_2$ so that they meet in a point $p$ in the outer face, and so that all the ends of $\sigma_1$ and $\sigma_2$ remain incident with the outer face (Figure 4). Let $\Sigma'$ be the obtained set of strings.

> Figure 4 Exterior-Meeting Step.

Seeking a contradiction, suppose that $G(\Sigma')$ has an obstruction $C$. Since $G(\Sigma)$ has no obstruction, $p \in V(C)$. Our contradiction will be to find three rainbows in $C$. Note that $p$ is a rainbow. To obtain a second rainbow, traverse $C$ starting from $p$ towards $a_1$. Let $d_1$ be the first vertex during our traversal that is not in the extended $\sigma_1$, and let $c_1$ be its neighbour in $\sigma_1$, one step before we reach $d_1$. Since $b_1$ has degree one, $c_1 \neq b_1$.

> Claim 13. The cycle $C$ has a rainbow included in the closed disk $\Delta_1$ bounded by $\sigma_1$ and the $a_1b_1$-arc of $O$ disjoint from $\sigma_2$. 

Proof. First, suppose that $d_1 \notin \Delta_1$. In this case, $c_1$ is a rainbow because otherwise there would be a string $\sigma$ that tangentially intersects $\sigma_1$ at $c_1$. Thus, if $d_1 \notin \Delta_1$, then $c_1$ is the desired rainbow.

Second, suppose that $d_1 \in \Delta_1$. Let $P_1$ be the path of $C$ starting at $c_1$, continuing on the edge $c_1d_1$, and ending at the first vertex we encounter in $\sigma_1$. Since the cycle $C'$ enclosed by $P_1 \cup \sigma_1$ is not an obstruction, there is one rainbow of $C'$ that is an interior vertex of $P_1$; this is the desired rainbow of $C$. This concludes the proof of Claim 13. ▷

Considering $\sigma_2$ instead of $\sigma_1$, Claim 13 yields a third rainbow in $C$ inside an analogous disk $\Delta_2$ disjoint from $\Delta_1$, contradicting that $C$ is an obstruction. Hence Lemma 12 holds. ▷

We iteratively apply the Disentangling Step, Face-Escaping Step or Exterior-Meeting Step without creating obstructions. Each step increases the number of pairwise intersecting strings in $\Sigma$ until we reach a stage where the strings are pairwise intersecting and all of them have their two ends in the unbounded face. From this we extend them into an arrangement of pseudolines. This concludes the proof of Theorem 2. ▷

5 Finding obstructions and extending strings in polynomial time

We start this section by describing an algorithm to detect obstructions. Henceforth, we assume that the input to the problem is the planarization $G(\Sigma)$ of an ordinary set of $s$ strings $\Sigma$. For the running-time analysis, we assume that $n$ and $m$ are the number of vertices and edges in $G(\Sigma)$, respectively. Since $G(\Sigma)$ is planar, $m = O(n)$. Moreover, if $\Sigma$ is pseudolinear, then $n \leq \binom{s}{2} + 2s = \binom{s+2}{2} - 1$. At the end of this section we explain how to extend $\Sigma$ (if possible) in polynomial time.

Recall that each string in $\Sigma$ receives a different colour; this induces an edge-colouring on $G(\Sigma)$ where each string is a monochromatic path. An outer-rainbow is a vertex $x \in V(G(\Sigma))$ incident with the outer face and for which the edges incident with $x$ have different colours.

Next we describe the basic operation in our obstruction-detecting algorithm.

![Figure 5 From Σ to Σ - x.](image)

Outer-rainbow deletion. Given an outer-rainbow $x \in V(G(\Sigma))$, the instance $G(\Sigma - x)$ is defined by: first, removing $x$ and the edges incident to $x$; second, suppressing the degree-2 vertices incident with edges of the same colour; and third, removing remaining degree-0 vertices (Figure 5 illustrates this process). Edge colours are preserved.

It is easy to verify that $G(\Sigma - x)$ is the planarization of an arrangement of strings. The colours removed by this operation are those belonging to strings that are paths of length 1 in $G(\Sigma)$ incident with $x$. Our obstruction-detecting algorithm relies on the following property:

(*** if $x$ is an outer-rainbow of $G(\Sigma)$, then there is an obstruction in $G(\Sigma)$ not including $x$ if and only if there is an obstruction in $G(\Sigma - x)$.)
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This property holds because cycles in $G(\Sigma) - x$ and in $G(\Sigma - x)$ are in 1-1 correspondence: two cycles correspond to each other if they are the same simple closed curve. This correspondence is obstruction-preserving.

Let us now describe the two subroutines in our algorithm. For this, we remark that an outer-rainbow of $G(\Sigma)$ is a rainbow for any cycle containing it.

- **Algorithm 1** Subroutine for detecting an obstruction through two outer-rainbows $x$ and $y$.

  1. Find a cycle $C$ through $x$ and $y$ whose edges are incident with the outer face of $G(\Sigma)$. If no such $C$ exists, output *No obstruction through $x$ and $y$*. Else, go to Step 2.
  2. Find whether there is a third outer-rainbow $z \in V(C) \setminus \{x, y\}$. If such $z$ exists, update $G(\Sigma) \leftarrow G(\Sigma - z)$ and go to Step 1. If no such $z$ exists, output $C$.

**Correctness and running-time of Algorithm 1:** If an obstruction through $x$ and $y$ exists, then $x$ and $y$ are in the same block (some authors use the term “biconnected component”). Since $x$ and $y$ are incident with the outer face, the outer boundary of the block containing $x$ and $y$ is the cycle $C$ from Step 1. This $C$ can be found by considering outer boundary walk $W$ of $G(\Sigma)$ and then by finding whether $x$ and $y$ belong to the same non-edge block of $W$. Finding $W$ is $O(m)$ and computing the blocks of $W$ via a DFS takes $O(m)$ time.

  In Step 2, if there is a third outer rainbow $z$ in $C$, then no obstruction through $x$ and $y$ contains $z$. Property (**)) justifies the update that takes $O(m)$ time. A full run from Step 1 to Step 2 takes $O(m)$. Moving from Step 2 to Step 1 occurs $O(n)$ times. Thus, the total time for Algorithm 1 is $O(mn) = O(n^2)$.

- **Algorithm 2** Subroutine for detecting an obstruction through a single outer-rainbow $x$.

  1. Find a cycle $C$ through $x$ whose edges are incident with the outer face of $G(\Sigma)$. If no such $C$ exists, output *No obstruction through $x$*. Else, go to Step 2.
  2. Find whether there is an outer-rainbow $y$ in $V(C) \setminus \{x\}$. If no such $y$ exists, output $C$.

  Else, apply Algorithm 1 to $x$ and $y$; if there is an obstruction $C'$ through $x$ and $y$, then output $C'$. Else, update $G(\Sigma) \leftarrow G(\Sigma - y)$ and go to Step 1.

**Correctness and running-time of Algorithm 2:** If $G(\Sigma)$ has an obstruction through $x$, then there is a non-edge block in $G(\Sigma)$ containing $x$. The outer boundary of this block is a cycle $C$ through $x$ having all edges incident with the outer face. As in Algorithm 1, Step 1 takes $O(m)$ time.

  Detecting the existence of $y$ in Step 2 is $O(m)$ because to detect rainbows in $C$, each edge incident with a vertex in $V(C)$ is verified at most twice. The update in Step 2 is justified by Property (**)). Since Step 2 may use Algorithm 1, Step 2 takes $O(n^2)$ time. As moving from Step 2 to Step 1 occurs $O(n)$ times, the total running-time for Algorithm 2 is $O(n^3)$.

  We are now ready for the algorithm to detect obstructions.

- **Algorithm 3** Detecting obstructions in $G(\Sigma)$.

  1. Find a cycle $C$ having all edges incident with the outer face. If no such $C$ exists, output *No obstruction*. Else, go to step 2.
  2. Find whether there is an outer rainbow $x \in V(C)$. If not, output $C$. Else apply Algorithm 2 to $x$; if there is an obstruction $C'$ through $x$, output $C'$. Else, update $G(\Sigma) \leftarrow G(\Sigma - x)$ and go to Step 1.
Correctness and running-time of Algorithm 3: If $G(\Sigma)$ has an obstruction, then it has a non-trivial block whose outer boundary is a cycle $C$ as in Step 1. As before, $C$ and $x$ as in Step 2 can be found in $O(m)$ steps. If $C$ has not outer rainbow $x$, then $C$ is an obstruction; Property (**) justifies the update in Step 2.

Since Step 2 may use Algorithm 2, a full run of Steps 1 and 2 takes $O(n^3)$ time. Since Step 2 goes to Step 1 $O(n)$ times, the running-time of Algorithm 3 is $O(n^4)$.

Algorithm 3 and the constructive proof of Theorem 2 imply the following result (proved in the full version of this paper [4]).

Theorem 14. There is a polynomial-time algorithm to recognize and extend an ordinary set of strings that are extendible to an arrangement of pseudolines.

6 Concluding remarks

In this work we characterized in Theorem 2 sets of strings that can be extended into arrangements of pseudolines. Moreover, we showed that the obstructions to pseudolinearity can be detected in $O(n^4)$ time, where $n$ is the number of vertices in the planarization of the set of strings.

An easy consequence of Theorem 2 is the following (presented before as Theorem 1). We prove this result in the full version of this paper [4].

Theorem 15. Let $D$ be a non-pseudolinear good drawing of a graph $H$. Then there is a subset $S$ of edge-arcs in $\{D[e]: e \in E(H)\}$, such that each $\sigma \in S$ has a substring $\sigma' \subseteq \sigma$ for which $\bigcup_{\sigma \in S} \sigma'$ is one of the drawings represented in Figure 1.

Theorem 2 can also be applied to find a short proof that pseudolinear drawings of $K_n$ are characterized by forbidding the $B$-configuration (see Theorem 2.5.1 in [3]). This implies the characterizations of pseudolinear drawings of $K_n$ presented in [2, 5, 6].

A drawing is stretchable if it is homeomorphic to a rectilinear drawing. There are pseudolinear drawings that are not stretchable. For instance, consider the Non-Pappus configuration in Figure 6. Nevertheless, as an immediate consequence of Thomassen’s main result in [16], pseudolinear and stretchable drawings are equivalent, under the assumption that every edge is crossed at most once.

Figure 6 Non-Pappus configuration.

Corollary 16. A drawing $D$ of a graph in which every edge is crossed at most once is stretchable if and only if it is pseudolinear.

Proof. If a drawing $D$ is stretchable then clearly it is pseudolinear. To show the converse, suppose that $D$ is pseudolinear. Then $D$ does not contain any obstruction, and in particular, neither of the $B$- and $W$-configurations in Figure 1 occurs in $D$. This condition was shown in [16] to be equivalent to being stretchable. ▷
One can construct more general examples of pseudolinear drawings that are not stretchable by considering non-stretchable arrangements of pseudolines. However, such examples seem to inevitably have some edge with multiple crossings. This leads to a natural question.

**Question 17.** Is it true that if $D$ is a pseudolinear drawing in which every edge is crossed at most twice, then $D$ is stretchable?

We believe that there are other instances where pseudolinearity characterizes stretchability of drawings. A drawing is near planar if the removal of one edge produces a planar graph. One instance, is the following result by Eades et al. that can be translated to the language of pseudolines:

**Theorem 18 ([9]).** A drawing of a near-planar graph is stretchable if and only if the drawing induced by the crossed edges is pseudolinear.

**References**