The extrinsic nature of the Hausdorff distance of optimal triangulations of manifolds

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Abstract

Fejes Tóth [5] and Schneider [9] studied approximations of smooth convex hypersurfaces in Euclidean space by piecewise flat triangular meshes with a given number of vertices on the hypersurface that are optimal with respect to Hausdorff distance. They proved that this Hausdorff distance decreases inversely proportional with $m^{2/(d-1)}$, where m is the number of vertices and d is the dimension of Euclidean space. Moreover the proportionality constant can be expressed in terms of the Gaussian curvature, an intrinsic quantity. In this short note, we prove the extrinsic nature of this constant for manifolds of sufficiently high codimension. We do so by constructing an family of isometric embeddings of the flat torus in Euclidean space.

1 Introduction

In [5] Fejes Tóth introduced inscribed triangulations approximating convex surfaces in $\mathbb{R}^3$ optimally and the 'Approximierbarkeit' (approximation parameter A_2). By a triangulation we shall mean a geometric realization of a simplicial complex in Euclidean space homeomorphic to the surface, that is piecewise linear in ambient space. From now on we take a simplicial complex to mean the geometric realization.

Optimal triangulations T_m with m vertices are triangulations which minimize the Hausdorff distance between the surface and the simplicial complex when this simplicial complex ranges over the space of triangulations with m vertices. We always assume that the vertices lie on the surface.

The Hausdorff distance between two subsets X and Y in a Euclidean space of arbitrary but fixed dimension d is defined as:

$$d_H(X,Y) = \max\{\sup_{x\in X}\inf_{y\in Y}|x-y|, \sup_{y\in Y}\inf_{x\in X}|x-y|\},$$

where $|x-y|$ denotes the standard Euclidean distance of x and y. The one-sided Hausdorff distance from X to Y is given by

$$d_H^o(X,Y) = \sup_{x\in X}\inf_{y\in Y}|x-y|.$$

The inverse of the asymptotic value of the product of the number of vertices and the Hausdorff distance is referred to as the *Approximierbarkeit* A_2.

Let K be the Gaussian curvature of the surface $\Sigma \subset \mathbb{R}^3$, and let dVol denote the volume (area) form on the surface. Fejes Tóth [5] gave the expression

$$\frac{1}{A_2} = \lim_{m\to\infty} d_H(\Sigma, T_m)m \geq \frac{1}{\sqrt{27}}\int_\Sigma \sqrt{K}\mathrm{dVol}, \qquad (1)$$

for the Approximierbarkeit for convex surfaces in three dimensional Euclidean space.

Schneider [9] generalized the discussion of Fejes Tóth to convex hypersurfaces (Σ) in Euclidean space of arbitrary dimension. The formula for A_{d-1}, derived by Schneider reads

$$\begin{aligned}\frac{1}{A_{d-1}} &= \lim_{m\to\infty} m^{2/(d-1)} d_H(\Sigma, T_m) \\ &= \frac{1}{2}\left(\frac{\theta_{d-1}}{\kappa_{d-1}}\int_\Sigma \sqrt{K(x)}\mathrm{dVol}\right)^{2/(d-1)}, \qquad (2)\end{aligned}$$

where $\kappa_d = \pi^{d/2}/\Gamma(1+d/2)$ is the volume of the d-dimensional unit ball, θ_d the covering density of the ball in d-dimensional space, dVol the volume form and K the Gaussian curvature. The covering density is defined as the infimum of the density over all coverings of, in this case, Euclidean space by the Euclidean unit ball, see for example [8]. The density of a cover $U = \{U_i\}$ of a compact measurable space with volume form dVol by a finite number of sets U_i is defined as follows: Let f_U be the integer valued function whose value $f_U(x)$ at a point x is the number of sets U_i such that $x \in U_i$. The density for the covering U is

$$\frac{\int f_U \,\mathrm{dVol}}{\int \mathrm{dVol}}.$$

Formula (2) is *intrinsic* in nature, because the Gaussian curvature is intrinsic. Generally, we call a quantity intrinsic if it depends only on the geometry of the surface or manifold itself. On the other hand a quantity is called extrinsic if depends on the embedding in the ambient space. If we for example consider a topological circle or loop in the plane, the length of the circle is intrinsic, while the curvature is not. This is because one can deform the loop without changing the distances

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between the points on the loop (as measured along the loop).

To make our statement concerning (2) more precise we note that the Gaussian curvature is invariant under isometry if $d-1$ is even and invariant up to sign if $d-1$ is odd, see [11, Chapter 7, Proposition 24]. It is clear that here the positive sign is the relevant one, because expression and thus the root $\sqrt{K}$ must be real.[1]

We will show that the intrinsic nature of the approximation parameter is particular to low co-dimension, by giving a sequence of isometric embeddings $E_k(\Sigma)$ of a surface Σ such that $\frac{1}{A_2}(E_k(\Sigma))$ tends to infinity with k. This means that there is no intrinsic quantity that can bound $\frac{1}{A_2}(E_k(\Sigma))$.

This makes heuristically some sense because the rigidity of a manifold disappears if the codimension of the embedding is sufficiently high, as was noted by Nash in [6]. In the setting of Nash, rigidity concerns metric preserving perturbations of the embedding. Nash proved that a compact n-manifold with a C^k Riemannian metric has a C^k isometric embedding in any small volume of Euclidean $(n/2)(3n+11)$-space, provided $3 < k \leq \infty$. So roughly speaking, one can squash a manifold in a small volume without affecting the intrinsic metric, but this would lead to wrinkles (and a build-up of extrinsic curvature).

This is the complete opposite of manifolds embedded in Euclidean space of codimension one, where manifold with non-zero curvature are embedded rigidly. Rigidity here means that an isometric embedding is unique up to Euclidean motions. Euclidean motions are generated by rotations and translations. We refer to Spivak [10, 12] for an overview of results on rigidity.

Our interest in the extrinsic nature was raised by the upper bounds on

$$\lim_{m\to\infty} d_H(M, T_m^o) m^{n/2},$$

where M is an n-dimensional manifold embedded in Euclidean space and T_m^o denotes an optimal triangulation of M. These bounds have been discussed in the Master's thesis of David de Laat [3]. Similar upper bounds were the topic of, among others[2], Chen, Sun and Xu[2]. These authors studied the L_p norm of the difference between a function and a linear approximation of this function. The bounds in [3] and [2] are defined in terms of the Hessian and thus extrinsic in nature. Our result below, gives us that the extrinsic nature of the bounds is unavoidable:

Theorem 1 *Let M be a Riemannian surface, then there is generally no function $f(g, \partial g, \dots)$ which depends only on the metric and all its derivatives and a constant $\tilde{c}$ such that*

$$\lim_{m\to\infty} d_H(T_m, E(M)) m \leq \tilde{c} \int_M f \, \mathrm{d}\,Vol,$$

where $E(M)$ denotes the embedding of the manifold in Euclidean space and $\mathrm{d}\,Vol$ the volume form.

We prove the theorem by constructing an explicit example of a family of embeddings, which we'll describe in detail in the next section.

2 The construction of a sequence of embeddings

We consider a family of isometric embeddings $E : S^1 \times S^1 \to \mathbb{R}^n$ of the flat torus, whose members are discriminated by the index $k \in \mathbb{Z}_{\geq 1}$. We write

$$\lim_{m\to\infty} d_H(E_k, T_m) m = c_{E_k},$$

where E_k indicates a member of the family of isometric embeddings of $S^1 \times S^1$, T_m is an optimal triangulation with m vertices that lie on E_k. c_{E_k} is a real number depending on E_k. For the family of embeddings, we construct we have

$$\lim_{k\to\infty} c_{E_k} = \infty.$$

To simplify the calculations we focus[3] on embeddings in $\mathbb{R}^8$. We shall study the family of embeddings of the flat torus E_k parameterized by $k \in \mathbb{Z}_{\geq 1}$:

$$\begin{aligned} \mathcal{E}_k(\theta, \varphi) = (\, & \cos(\theta), \sin(\theta), \cos(k\theta)/k, \sin(k\theta)/k, \\ & \cos(\varphi), \sin(\varphi), \cos(k\varphi)/k, \sin(k\varphi)/k) \\ E_k = \{ & E_k(\theta, \varphi) \mid \theta \in [0, 2\pi], \varphi \in [0, 2\pi]\}. \end{aligned}$$

Note that $\psi \mapsto (\cos(\psi), \sin(\psi), \cos(k\psi)/k, \sin(k\psi)/k)$, is an embedding of the circle in $\mathbb{R}^4$. This makes E_k an embedded flat torus.

Because E_k contains no straight line segments, we see that

Lemma 2 *For each (fixed) k the edge length of each edge in a triangulation T_m tends to zero as $d_H(T_m, E_k)$ tends to zero.*

Proof. Suppose that there is a subsequence $T_{m(l)}$ for which the length of edges $e_{m(l)} \in T_{m(l)}$ does not tend to zero. Without loss of generality we can assume (by chosing a convergent subsequence) that $e_{m(l)}$ converges to a limit line element, whose length by assumption is not zero. Because we assume that the Hausdorff distance $d_H(T_m(l), E_k)$ tends to zero, this line element lies within E_k, which contradicts the fact that E_k contains no straight lines. $\square$

[1] We know that for a large class of negatively curved surfaces in $\mathbb{R}^3$ that $\frac{1}{A_2}$ is proportional to $\int \sqrt{|K|} \mathrm{d}A$, see [1, 7, 13].

[2] The introduction of [2] offers an extensive literature overview.

[3] It should be possible to prove the result for embeddings in $\mathbb{R}^4$, where the flat torus is not rigid, but the calculations would be significantly more difficult.

Because of Lemma 2 we may locally approximate the surface. In particular the tangent plane of the surface in the neighbourhood of a triangle is asymptotically well defined, because the triangle becomes small. Secondly, we may employ the natural group action of $(\mathrm{SO}(2))^4$ on the ambient space $\mathbb{R}^8$ to shift a given point on the torus to the origin. This means that we can approximate the surface, parametrized by $\mathcal{E}_k$, locally by

$$(1-\theta^2/2, \theta, 1-\theta^2/(2k), \theta, 1-\varphi^2/2, \varphi, 1-\varphi^2/(2k), \varphi),$$

where (θ, φ) are near the origin, and thus through a translation by

$$\Sigma_k(\theta,\varphi) \simeq (-\theta^2/2, \theta, -\theta^2/(2k), \theta, -\varphi^2/2, \varphi, -\varphi^2/(2k), \varphi).$$

Furthermore we may assume that the vertices of a triangle are $\Sigma_k(0,0) = 0 \in \mathbb{R}^8$, $\Sigma_k(\theta_1, \varphi_1)$, $\Sigma_k(\theta_2, \varphi_2)$.

We shall employ techniques similar to the ones employed by Fejes Tóth [5] to find a lower bound for $\frac{1}{A_2}$. To be precise we fix the one-sided Hausdorff distance for a family of triangles and search for the triangle in the family with the largest area. The area of surface divided by the area of the largest triangle in the family will give a bound on the number of triangles needed in a triangulation that attains the fixed Hausdorff distance.

Because the number of triangles ($\tilde{m}$), edges (e) and vertices (m) are related by the fact that every triangle has three edges and each edge is shared by two triangles as well as the formula for the Euler characteristic $\chi = \tilde{m} - e + m$ a bound on $\frac{1}{A_2}$ follows. Here we are only interested in (rough) lower bounds, so it suffices to fix some lower bound on the Hausdorff distance and then determine some upper bound on the area of the triangles in a triangulation satisfying this bound.

To be able to fix a bound on the Hausdorff distance we calculate the following:

Lemma 3 *The one-sided Hausdorff distance d_H^o of a triangle on the surface parametrized by $\Sigma_k(\theta,\varphi)$ with vertices $(0,0)$, (θ_1,φ_1) and (θ_2,φ_2) satisfies:*

$$d_H^o \geq \eta = \frac{1}{8}\sqrt{1+k^2}\max\left\{\sqrt{\theta_1^4+\varphi_1^4}, \sqrt{\theta_2^4+\varphi_2^4}\right\} + \mathcal{O}(|(\theta,\varphi)|^2).$$

Proof. A point p on the triangle with vertices $(0,0)$, (θ_1,φ_1) and (θ_2,φ_2) will be given as $p = \Sigma_k(\theta_1,\varphi_1)\lambda_1 + \Sigma_k(\theta_2,\varphi_2)\lambda_2$, with $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0, 1-\lambda_1]$. We now want to find the point on the surface $\Sigma_k(\theta_c,\varphi_c)$ which is closest to p. This point is determined my the following equations:

$$\partial_\theta |p - \Sigma_k(\theta,\varphi)|^2 = 0$$
$$\partial_\varphi |p - \Sigma_k(\theta,\varphi)|^2 = 0.$$

It is not difficult to verify that $\theta_c \simeq \theta_1\lambda_1 + \theta_2\lambda_2$ and $\varphi_c \simeq \varphi_1\lambda_1 + \varphi_2\lambda_2$, where $\simeq$ denotes equality up to linear order in θ_i, φ_i. This means that the distance between a point on the triangle and the surface is approximately given by

$$\begin{aligned} &\|\Sigma_k(\theta_1,\varphi_1)\lambda_1 + \Sigma_k(\theta_2,\varphi_2)\lambda_2 \\ &\qquad - \Sigma_k(\theta_1\lambda_1+\theta_2\lambda_2, \varphi_1\lambda_1+\varphi_2\lambda_2)\| \\ &= \Bigg|\Bigg((\theta_1\lambda_1+\theta_2\lambda_2)^2/2 - \frac{\theta_1^2}{2}\lambda_1 - \frac{\theta_2^2}{2}\lambda_2, 0, \\ &\quad k(\theta_1\lambda_1+\theta_2\lambda_2)^2/2 - k\frac{\theta_1^2}{2}\lambda_1 - k\frac{\theta_2^2}{2}\lambda_2, 0, \\ &\quad (\varphi_1\lambda_1+\varphi_2\lambda_2)^2/2 - \frac{\varphi_1^2}{2}\lambda_1 - \frac{\varphi_2^2}{2}\lambda_2, 0, \\ &\quad k(\varphi_1\lambda_1+\varphi_2\lambda_2)^2/2 - k\frac{\varphi_1^2}{2}\lambda_1 - k\frac{\varphi_2^2}{2}\lambda_2, 0\Bigg)\Bigg|. \end{aligned}$$

For the choice $\lambda_1 = 1/2$, $\lambda_2 = 0$ and $\lambda_1 = 0$, $\lambda_2 = 1/2$ this yields $\sqrt{1+k^2}\sqrt{\theta_1^4+\varphi_1^4}/8$ and $\sqrt{1+k^2}\sqrt{\theta_2^4+\varphi_2^4}/8$ respectively, so

$$d_H \geq \eta = \frac{1}{8}\sqrt{1+k^2}\max\left\{\sqrt{\theta_1^4+\varphi_1^4}, \sqrt{\theta_2^4+\varphi_2^4}\right\} + \mathcal{O}(|(\theta,\varphi)|^2).$$

□

From Lemma 3 we can conclude that

$$\frac{8}{\sqrt{1+k^2}}\eta \geq \theta_1^2, \theta_2^2, \varphi_1^2, \varphi_2^2.$$

On the other hand the area of the triangle is approximately equal to

$$|\varphi_1\theta_2 - \theta_1\varphi_2|/2.$$

So the area of a triangle is bounded from above by

$$\frac{4}{\sqrt{1+k^2}}\eta. \tag{3}$$

Let us denote by π_{E_k} the closest point projection onto E_k. Although it seems intuitively clear that the projection π_{E_k} of T_m to E_k is surjective for sufficiently small Hausdorff distance, it is not so easy to prove. We will use that any point in E_k must be at most a distance $d_H(T_m, E_k)$ from a (projected) triangle $\pi_{E_k}(t)$ in $\pi_{E_k}(T_m)$. We furthermore recall that Theorem 4.8(8) of [4] gives us that the closest point projection on a set of positive reach $\mathcal{S}$ is a Lipschitz map with Lipschitz constant

$$\frac{\mathrm{rch}(\mathcal{S})}{\mathrm{rch}(\mathcal{S}) - \delta},$$

where rch denotes the reach and δ an upper bound on the distance of the points to the set, that is the distance between the medial axis and the set itself. As the Hausdorff distance tends to zero the Lipschitz constant of the projection tends to 1. Using this we see that the area of the d_H-neighbourhood of a triangle $\pi_{E_k}(t)$ equals the area of the triangle t itself plus the length of the boundary times d_H plus higher order terms. Using the estimates in Lemma 3 we find that the area of the neighbourhood is $\text{Area}(t) + \mathcal{O}(\text{Area}(t)^{1/2} d_H)$ or equivalently $\text{Area}(t) + \mathcal{O}(d_H^{3/2})$. So up to leading order the bound of (3) on the area holds even after projecting and with a safe margin.

As we already noted above, the number of triangles $\tilde{m}$ in a triangulation is bounded from below by

$$\text{Area}(t_{\max})\tilde{m} \gtrsim \text{Area}(E_k),$$

where $\gtrsim$ is used to suppress terms that are not of leading order in the Hausdorff distance, E_k denotes the embedding of the surface and $t_{\max}$ denotes the biggest triangle in the triangulation. These considerations give us

$$\begin{aligned} d_H m &\gtrsim \eta\, m \\ &\gtrsim \eta \frac{\text{Area}(E_k)}{\text{Area}(\triangle)} \\ &\gtrsim \eta \frac{(4\pi)^2}{4\eta/\sqrt{1+k^2}} \\ &= 4\pi^2\sqrt{1+k^2} \end{aligned}$$

This implies that

$$\lim_{m\to\infty} d_H(T_m, E_k)m \geq 4\pi^2\sqrt{1+k^2}.$$

The result is summarized in the main theorem

Theorem 1 *Let M be a Riemannian surface, then there is generally no function $f(g, \partial g, \ldots)$ which depends only on the metric and all its derivatives and a constant $\tilde{c}$ such that*

$$\lim_{m\to\infty} d_H(T_m, E(M))m \leq \tilde{c}\int_M f \,\mathrm{d}\, Vol,$$

where $E(M)$ denotes the embedding of the manifold in Euclidean space and $\mathrm{d}\,Vol$ the volume form.

In this theorem we could have absorbed $\tilde{c}$ in $f(g)$. However, we have chosen this form to mimic the traditional form of the result of Fejes Tóth [5] and Schneider [9]. The generalization of the above theorem to manifolds of arbitrary dimension is straightforward, because one can take cross product of our example with any other manifold and find the same result.

3 Open question: rigidity

If the embedding of a manifold M is rigid then

$$\lim_{m\to\infty} m d_H(M, T_m)^{(n-1)/2},$$

where again T_m is optimal, must only depend on intrinsic quantities. This is because any two embeddings are the same up to Euclidean motions and thus the intrinsic geometry determines the geometry of the embedding completely.

What the converse statement should be is not so clear. For example the cylinder with boundaries $S^1 \times [a, b]$ is non-rigid, while the limit of $md_H(M, T_m)$ is independent of embedding, albeit zero. It would be interesting to understand the exact relation between rigidity and the asymptotic behavior with respect to m of $d_H(M, T_m)$ better. Results in this direction could also provide a different perspective on combinatorial rigidity.

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