

# Hanani-Tutte for Radial Planarity

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**Abstract.** A drawing of a graph  $G$  is *radial* if the vertices of  $G$  are placed on concentric circles  $C_1, \dots, C_k$  with common center  $c$ , and edges are drawn *radially*: every edge intersects every circle centered at  $c$  at most once.  $G$  is *radial planar* if it has a radial embedding, that is, a crossing-free radial drawing. If the vertices of  $G$  are ordered or partitioned into ordered levels (as they are for leveled graphs), we require that the assignment of vertices to circles corresponds to the given ordering or leveling.

We show that a graph  $G$  is radial planar if  $G$  has a radial drawing in which every two edges cross an even number of times; the radial embedding has the same leveling as the radial drawing. In other words, we establish the weak variant of the Hanani-Tutte theorem for radial planarity. This generalizes a result by Pach and Tóth.

## 1 Introduction

In a *leveled* graph every vertex is assigned a level in  $\{1, \dots, k\}$ . We can capture the leveling of the graph visually, by placing the vertices on parallel lines or concentric circles corresponding to the levels of  $G$ . To further emphasize the levels, we can require that edges respect the levels in the sense that edges must lie between the levels of their endpoints, and be monotone in the sense that they intersect any line (circle) parallel to (concentric with) the chosen lines (circles) at most once. If we choose lines, we obtain the concept of *level-planarity*; for circles we get *radial (level) planarity*.

Radial planarity was introduced by Bachmaier, Brandenburg and Forster [1] as a generalization of level-planarity [6]. Radial layouts are a popular visualization tool (see [7] for a recent survey); early examples of radial graph layouts can be found in the literature on sociometry [13]. Bachmaier, Brandenburg and Forster [1] showed that radial planarity can be tested, and an embedding can be found, in linear time. Their algorithm is based on a variant of PQ-trees [2] and is rather intricate. It generalizes an earlier linear time algorithm for level-planarity

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testing by Jünger and Leipert [12]. In this paper, we take the first step toward an alternative algorithm for radial planarity testing via a Hanani-Tutte style characterization.

The classical Hanani-Tutte theorem [5,20] states that a graph is planar if and only if it can be drawn in the plane so that every two independent edges cross an even number of times. A particularly nice algorithmic consequence of this result is that it reduces planarity testing to solving a system of linear equations (of polynomial size) over  $\mathbb{Z}_2$ , a purely algebraic problem which can be solved in polynomial time.

If we could show that a leveled graph  $G$  is radial planar if it has a radial drawing (respecting the leveling) in which every two *independent* edges cross an even number of times, we would have a new, very simple, polynomial-time algorithm for radial planarity. We conjecture that this (strong) Hanani-Tutte characterization of radial planarity is true, and take the first step toward this result: a *weak* Hanani-Tutte theorem. A weak variant of the Hanani-Tutte theorem makes the stronger assumption that *every* two edges cross an even number of times. Often, this leads to stronger conclusions. For example, it is known that if a graph can be drawn in a surface so that every two edges cross evenly, then the graph has an embedding on that surface with the same rotation system, i.e. the cyclic order of ends at each vertex remains the same [3,16].

Our main result, proved in Section 3, is the following theorem:

**Theorem 1.** *Suppose a leveled graph  $G$  has a radial drawing in which every two edges cross an even number of times. Then  $G$  has a radial embedding with the same rotation system. (All drawings respect the given leveling.)*

Theorem 1 implies a polynomial time algorithm for the radial planarity testing of a leveled graph  $G$  if a combinatorial embedding (rotation system) of  $G$  is fixed. This algorithm (sketched in Section 4) is based on solving a system of linear equations over  $\mathbb{Z}_2$ , see also [19, Section 1.4]. Thus, our algorithm runs in time  $O(|V(G)|^{2\omega})$ , where  $O(n^\omega)$  is the complexity of multiplication of two square  $n \times n$  matrices. Since our linear system is sparse, it is also possible to use Wiedemann’s randomized algorithm [21], with expected running time  $O(n^4 \log n^2)$  in our case.

*Remark 1.* While we do not know whether the (strong) Hanani-Tutte characterization of radial planarity holds, it can still be used for an algorithmic solution of the radial planarity problem, in the following sense: one can write a polynomial-time algorithm which—given leveled graph  $G$ —either returns a radial planar embedding of  $G$ , states that  $G$  is not radial planar, or stops with “don’t know”. If the algorithm outputs one of the first two answers, it is correct. If the strong Hanani-Tutte theorem for radial planarity is true, then the third answer will not occur. The details of how this can be done have been explained in a paper by Gutwenger, Mutzel, and Schaefer [11], where such an algorithm was successfully implemented for  $c$ -planarity. ■

Theorem 1 is a generalization of a weak variant of the Hanani-Tutte theorem for level-planarity<sup>1</sup>, first proved by Pach and Tóth [14,9]. The full Hanani-Tutte theorem for level-planarity was established only more recently [9], and it led to a quadratic time level-planarity test. A computational study of Chimani and Zeranski [4] of various algorithms for upward planarity testing (an NP-complete problem related to level-planarity), showed that the algorithm based on the Hanani-Tutte characterization of level-planarity performs very well in practice (it beats all other algorithms in nearly all scenarios).

Hanani-Tutte style characterizations have also been established for partially embedded planar graphs, several classes of simultaneously embedded planar graphs [18], and two-clustered graphs [8]. The family of counterexamples in [8, Section 4] shows that a straightforward variant of the Hanani-Tutte theorem for clustered graphs with more than two clusters fails. Gutwenger et al. [11] showed that by using the reduction from [18], this counterexample can be turned into a counterexample for a variant of the Hanani-Tutte theorem for two simultaneously embedded planar graphs [18, Conjecture 6.20]. For higher-genus (compact) surfaces, the weak variant is known to hold in all surfaces [3,17], while the strong variant is known for the projective plane only [15]. It remains an intriguing open problem whether the strong Hanani-Tutte theorem holds for closed surfaces other than the sphere and projective plane.

## 2 Terminology

For the purposes of this paper, graphs may have multi-edges, but no loops. An *ordered* graph  $G = (V, E)$  is a graph whose vertices are equipped with a total order  $v_1 < v_2 < \dots < v_n$ . We consider an ordered graph a special case of a *leveled* graph, in which every vertex of  $G$  is assigned a *level*, a number in  $\{1, \dots, k\}$  for some  $k$ . The leveling of the vertices induces a natural ordering of the vertices.

For convenience we represent radial drawings as drawings on a (standing) cylinder. Intuitively, imagine placing a cylindrically-shaped mirror in the center of a radial drawing as described in the introduction.<sup>2</sup>

The *cylinder*  $\mathcal{C}$  is  $\mathbb{S}^1 \times I$ , where  $\mathbb{S}^1$  is a unit circle and  $I$  is the unit interval  $[0, 1]$ . Thus, a *point* on  $\mathcal{C}$  is a pair  $(s, i)$ , where  $s \in \mathbb{S}^1$  and  $i \in I$ . The *projection* of  $\mathcal{C}$  to  $\mathbb{S}^1$ , or  $I$ , maps  $(s, i) \in \mathcal{C}$  to  $s$ , or  $i$ . We denote a projection of a point or a subset  $\alpha$  of  $\mathbb{S}^1 \times I$  to  $I$  by  $I(\alpha)$ . The *winding number* of a closed curve on a cylinder is the number of times the projection to  $\mathbb{S}^1$  of the curve winds around  $\mathbb{S}^1$ , i.e., the number of times the projection passes through an arbitrary point of  $\mathbb{S}^1$  in the counter clockwise sense minus the number of times the projection passes through the point in the clockwise sense. A closed curve (or a cycle in a graph) on a cylinder is *essential* if it has an odd winding number.

<sup>1</sup> The result is stated for  $x$ -monotonicity, the special case of level-planarity in which every level contains a single vertex. As we will see below, this special case is equivalent to the general case.

<sup>2</sup> Search for “cylindrical mirror anamorphoses” on the web for many cool pictures of this transformation.

With this, a *radial drawing* of  $G$  is a drawing of  $G$  on  $\mathcal{C}$  such that the projection to  $I$  of every edge is injective (i.e., an edge does not “turn back”) and for every pair of vertices  $v_i < v_j$  we have  $I(v_i) < I(v_j)$ . We also speak of an edge being *radial* when it satisfies this condition. In a radial drawing an *upper edge* and *lower edge*, respectively, at  $v$  is an edge incident to  $v$  for which  $\min(I(e)) = I(v)$  and  $\max(I(e)) = I(v)$ . A vertex  $v$  is a *sink* (*source*), if  $v$  has no upper (lower) edges. In order to avoid some inconvenient situation we assume that  $I(G)$  is contained in the interior of  $I$ .

The *rotation* at a vertex in a drawing (on any surface) of a graph is the cyclic order of the ends of edges incident to the vertex in the drawing. The *rotation system* is the set of rotations at all the vertices in the drawing. In the case of radial drawings the *upper* (*lower*) rotation at a vertex  $v$  is the linear order of the end pieces of the upper (lower) edges in the rotation at  $v$  starting with the direction corresponding to the clockwise orientation of  $\mathbb{S}^1$ . The rotation at a vertex in a radial drawing is completely determined by its upper and lower rotation. The *rotation system* of a radial drawing is the set of the upper and lower rotations at all the vertices in the drawing.

In what follows we consider drawings of  $G$  in the (Euclidean) plane or on a cylinder. Thus, every embedded cycle of  $G$  is *separating*, i.e. its complement in the ambient space of  $G$  has two components. Also, the complement of any closed curve (possibly with self-crossings) can be *two-colored* so that connected regions each get one color and neighboring regions receive opposite colors.

A drawing of  $G$  is *even* if every two edges in the drawing cross an even number of times. After a sufficiently generic continuous deformation of a drawing of  $G$  the parity of the number of crossings between a pair of edges changes only when an edge passes through a vertex during the deformation. We call this event an *edge-vertex switch*. In particular, when an edge  $e$  passes through a vertex  $v$  the parity of the number of crossings between  $e$  and every edge incident to  $v$  changes.

### 3 Weak Hanani-Tutte for Radial Drawings

In this section, we prove Theorem 1, the weak Hanani-Tutte theorem for radial planarity. We claim that it will be sufficient to restrict ourselves to the special case in which every level of  $G$  contains a single vertex, an *ordered* graph.

**Theorem 2.** *Suppose the ordered graph  $G$  has an even radial drawing. Then  $G$  has a radial embedding with the same rotation system, and the winding parity of every cycle remains the same.<sup>3</sup> (All drawings respect the ordering.)*

The reduction of Theorem 1 to Theorem 2 is based on the same construction used in [9, Section 4.2] to reduce level-planarity to  $x$ -monotonicity: Suppose we are given an even radial drawing of a leveled graph  $G$ . If any level of  $G$  contains

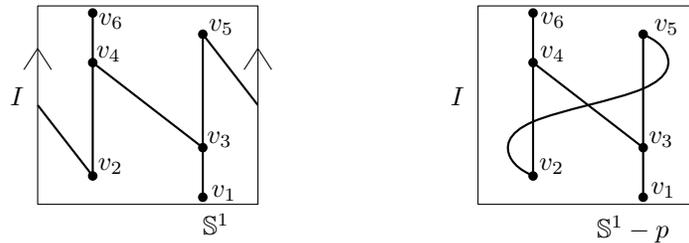
<sup>3</sup> The claim about the invariance of the parity of the winding number of every cycle in Theorem 2 is a consequence of the preservation of the rotation system (a proof will be included in the journal version).

more than one vertex, we do the following: if any vertex at that level is a source or a sink, we add a crossing-free edge on the empty side of that vertex. We place the new vertex at a new level, close to the current level we are working on. We now slightly perturb all the vertices of the current level so no two vertices are at the same level (without moving them past any of the new vertices we created). We can do so, while keeping all edges radial, and without introducing any crossings. Since the new vertices we added are at unique levels, we only perform the perturbation on the original levels. Call the resulting ordered graph  $G'$ . By Theorem 2,  $G'$  has a radial embedding with the same rotation system, and the winding number of every cycle unchanged. We can now move all perturbed vertices back to their original levels, the additional edges we added ensure that this is always possible.

We will make use of the weak Hanani-Tutte theorem for  $x$ -monotone graphs due to Pach and Tóth [14], reproved in [9].

**Theorem 3 (Pach, Tóth [14]).** *Suppose that  $G$  can be drawn so that edges are  $x$ -monotone and every two edges cross an even number of times. Then there exists an embedding of  $G$ , in which the vertices are drawn as in the given drawing of  $G$ , the edges are  $x$ -monotone, and the rotation system is the same.*

Figure 1 shows an example of a graph for which  $x$ -monotonicity and radial planarity differ. A radial embedding of a graph not admitting an  $x$ -monotone embedding, must contain an essential cycle.



**Fig. 1.** An instance of an ordered graph that admits a radial embedding but does not admit an  $x$ -monotone embedding. The left and the right edge of the rectangle are identified in the left part of the figure.

### 3.1 Working with Even Radial Drawings

Given a connected graph  $G$  with a rotation system, we can define a *facial walk* purely combinatorially by following the edges according to the rotation system (see, for example, [10, Section 3.2.6]). A vertex and edge, respectively, is *incident* to a face if it appears on its facial walk. By gluing disks to each facial walk, one obtains a surface with the graph embedded in it. In case the surface is a sphere,

each vertex occurs at most once on every facial walk of  $G$  if and only if  $G$  does not contain a cut-vertex. Any drawing of a graph  $G$  on an orientable surface defines a rotation system. For an even drawing of a connected graph  $G$  on the plane, the facial walks obtained from the rotation system correspond to a planar embedding. This is the weak Hanani-Tutte theorem [3,16]:

**Theorem 4.** *Let  $G$  be a graph. Suppose that  $G$  has an even drawing in the plane. Then  $G$  has an embedding in the plane with the same rotation system.*

Let  $v$  denote a vertex incident to a face  $f$ . Let  $e = uv$  and  $e' = vw$  denote a pair of consecutive edges on the facial walk of a face  $f$  in an embedding of a graph  $G$ . (Note that  $e$  and  $e'$  might be equal.) The edges  $e$  and  $e'$  define a *wedge* at  $v$  in  $f$  which is a portion of  $f$  in a small topological  $\epsilon$ -neighborhood of  $v$  between  $e$  and  $e'$ . By Theorem 4, any even drawing of  $G$  corresponds to its embedding with the same rotation system. Hence, every consecutive pair of edges  $e = uv$  and  $e' = vw$  in the rotation at  $v$  in an even drawing defines a wedge in the corresponding embedding of  $G$ . Thus, a face  $f$  in an even drawing of  $G$  is given by the set of wedges corresponding to  $f$  in the embedding of  $G$ . From the set of wedges representing a face  $f$  in an even drawing we obtain the facial walk  $W_f$  of  $f$ , since every two consecutive edges on  $W_f$  define a wedge of  $f$ . By slightly abusing the notation we will denote the facial walk  $W_f$  by  $f$ .

We consider a radial embedding of a connected graph  $G$ . Let a *maximum* (*minimum*) of a face  $f$  in the radial embedding of  $G$  be the maximum (minimum)  $v_i$  such that  $v_i$  is incident to  $f$ . Let a *local maximum* (*minimum*) of a face  $f$  be a vertex  $v$  incident to  $f$  such that  $v > u, w$  ( $v < u, w$ ), where  $u$  and  $w$  are the predecessor and successor of  $v$  on the facial walk of  $f$ . An *outer face* in a radial embedding is a face containing  $0 \times \mathbb{S}^1$  or  $1 \times \mathbb{S}^1$ . A face is an *inner face* if it is not an outer face.

Let  $f$  be a face in  $G$  given by an even radial drawing. The *boundary curve* of  $f$  is a closed curve traversing the facial walk of  $f$  in its close topological neighborhood, and at vertices passing through wedges in  $f$ . We naturally extend the notion of the winding number to a face defined as the winding number of its boundary curve. The *two-coloring* of  $f$  is the two-coloring of the complement of its boundary curve. A point in the complement of the boundary curve of  $f$  is in the *interior* (*exterior*) of  $f$  if it receives the same (opposite) color as a wedge in  $f$  when we two-color  $f$ . Note that in an even drawing all the wedges in a face have the same color. The outer face in an even radial drawing is a face having  $0 \times \mathbb{S}^1$  or  $1 \times \mathbb{S}^1$  in its interior. In an even radial drawing an outer face containing  $0 \times \mathbb{S}^1$  ( $1 \times \mathbb{S}^1$ ) is the *lower* (*upper*) outer face. If  $G$  has only one outer face  $f$ ,  $f$  is simultaneously the lower and upper outer face. A face that is not an outer face is an *inner face*. For a closed non-essential curve on the cylinder we define the *interior* (*exterior*) as the union of the connected components in its complement whose color is different (the same) as the color of  $0 \times \mathbb{S}^1$  in the two-coloring of its complement.

Let  $v$  be a local minimum or maximum of a face  $f$ . A wedge  $\omega$  at  $v$  in  $f$  in a radial drawing is *convex* (*concave*) if the angle bounding  $\omega$  in a small topological

neighborhood of  $v$  is convex and concave. Let  $C$  be a cycle in a radial drawing of  $G$ . We consider  $C$  to be a subgraph of  $G$  whose drawing is inherited from that of  $G$ . Now,  $C$  represents two faces. It is easy to see that  $C$  is non-essential if and only if in the two-coloring of (the complement of)  $C$  the concave wedge of  $C$  at the minimum and maximum of  $C$  receive the same color, i.e., are in the same face defined by  $C$ . Indeed, the parity of the number of crossings between a “vertical” path in  $\mathcal{C}$  joining  $0 \times \mathbb{S}^1$  with  $1 \times \mathbb{S}^1$ , and  $C$  equals to the winding number of  $C$  by the definition of the winding number of  $C$ .

In a radial embedding of a connected graph  $G$  we observe that a face  $f$  is an outer face if and only if  $v_n$  is its maximum, or  $v_1$  is its minimum, and the wedge in  $f$  at  $v_n$  or  $v_1$  is concave. Thus, either  $G$  has two outer faces one of which is lower and one of which is upper, or  $G$  has exactly one outer face. In the former, the boundary curve of the outer faces is essential. In the latter  $G$  does not contain any essential cycle. The same is true also in even radial drawings.

**Lemma 1.** *In an even radial drawing of a connected graph  $G$  at most one face can have a concave wedge at its maximum or minimum, which necessarily happens only at  $v_n$  or  $v_1$ . Consequently, either  $G$  has two outer faces one of which is lower and one of which is upper, or  $G$  has exactly one outer face.*

The next lemma simplifies the type of faces we have to deal with.

**Lemma 2.** *Let  $G$  be a connected graph. Suppose that  $G$  has an even radial drawing. Then we can augment the drawing of  $G$  by adding edges so that the resulting drawing is still even and radial, every face of  $G$  has at most two local minima and two maxima, and each outer face has exactly one local minimum and one local maximum.*

Proofs of previous lemmata will be contained in the journal version of the paper.

### 3.2 Proof of Theorem 2

A connected component  $G_1$  of  $G$  drawn radially is *essential* if it contains an essential cycle.

Given a graph  $G$  with a radial drawing, consider the augmentation of  $G$  that’s guaranteed by Lemma 2. If Theorem 2 is true for the augmented graph, then it is clearly true for  $G$  as well. Thus, it suffices to prove Theorem 2 for graphs of the form constructed in Lemma 2. Even more, we can restrict ourselves to connected components of the graph, as the following lemma shows (proof left to the journal version).

**Lemma 3.** *A counterexample to Theorem 2 with the smallest number of vertices is not disconnected.*

Before we turn to the proof of Theorem 2 we present one more tool which allows us to clear an arbitrary edge in an even radial drawing of crossings while

keeping the drawing radial and even. This is a slight extension of redrawing results we have used in previous papers [17, Figure 3]. A proof will be included in the journal version.

**Lemma 4.** *Let  $G$  denote a graph given by an independently even radial drawing. Let  $e$  denote an arbitrary edge of  $G$  crossing every other edge in  $G$  evenly. We can redraw the edges crossing  $e$  inside  $I(e) \times \mathbb{S}^1$  so that (i)  $e$  is crossing free, (ii) the resulting drawing remains even and radial, (iii) the rotation system and the points representing vertices are the same as in the given drawing, (iv) the parity of winding number remains the same for all cycles.*

*Proof (of Theorem 2).* For the sake of contradiction let  $G$  denote a minimal counterexample with respect to the number  $N$  of ordered pairs of vertices  $(u, v)$ ,  $u < v$ , of  $G$ , where  $u$  is a source and  $v$  is a sink. By Lemma 3, we assume that  $G$  is connected and by Lemma 2 we assume that each face of  $G$  contains at most two local minima and each outer face at most one local minimum. We construct a radial embedding  $\mathcal{D}$  of  $G$  inductively as  $\mathcal{D}_n$ , where every  $\mathcal{D}_i$ ,  $1 \leq i \leq n$ , is an even radial drawing of a graph  $G_i$  obtained from  $G$  by subdividing certain edges in their interior without altering its radial planarity such that  $\mathcal{D}_i$  is crossing free in  $[0, I(v_i)] \times \mathbb{S}^1$ . Since throughout the proof we keep the rotation system of  $G$  unchanged (after suppressing the subdividing vertices in  $G_i$ ), this contradicts the choice of  $G$ .

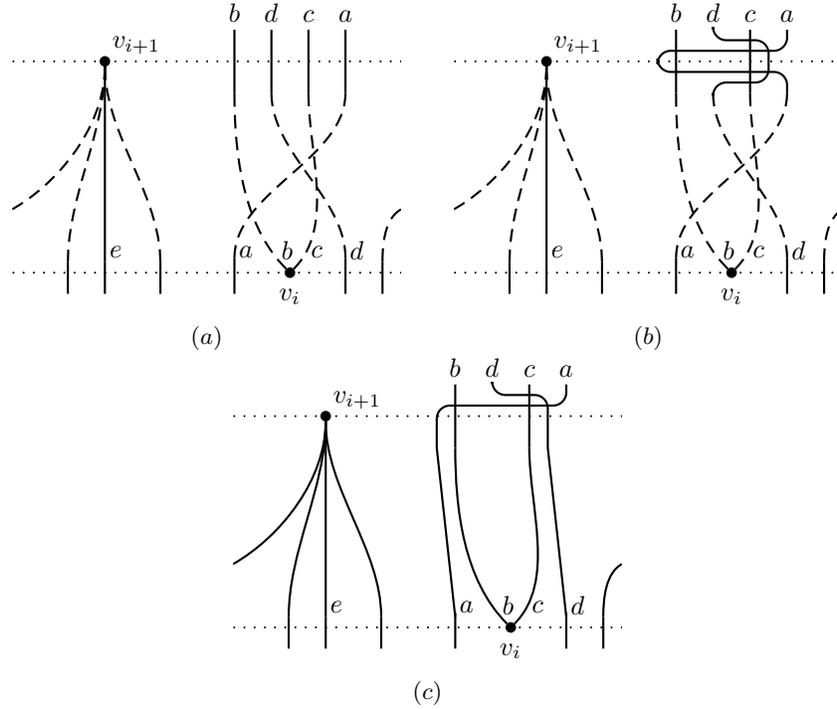
We proceed by constructing compatible cyclic orders  $\mathcal{O}_i$  given by the order of appearance of points in  $\mathcal{D}_i \cap (I(v_i) \times \mathbb{S}^1)$  along  $I(v_i) \times \mathbb{S}^1$ . The elements, i.e., points, in  $\mathcal{O}_i$ 's are denoted by the objects of  $G_i$ , i.e., edges and vertices, they belong to. By ‘‘compatible’’ we mean that for two consecutive orders  $\mathcal{O}_i$  and  $\mathcal{O}_{i+1}$  (after suppressing the subdividing vertices in  $G_i$ ) the order  $\mathcal{O}_{i+1}$  is obtained from  $\mathcal{O}_i$  using an auxiliary order  $\mathcal{O}'_i$  as follows. We obtain  $\mathcal{O}'_i$  from  $\mathcal{O}_i$  by replacing  $v_i$  with its upper edges. We obtain  $\mathcal{O}_{i+1}$  from  $\mathcal{O}'_i$  by replacing the lower edges of  $v_{i+1}$  appearing consecutively in  $\mathcal{O}'_i$  with  $v_{i+1}$  if  $v_{i+1}$  is not a source. Otherwise, we obtain  $\mathcal{O}_{i+1}$  from  $\mathcal{O}'_i$  by placing  $v_{i+1}$  between two edges bounding the face that contains the concave wedge at  $v_{i+1}$ .

In the base case there is nothing to prove. We just put  $G_1 := G$ . For  $i+1 > 1$ , we distinguish two cases depending on whether  $v_{i+1}$  is a source or not. We work in  $\mathcal{D}_i$  from the induction hypothesis. We subdivide in  $G_i$  every edge  $e'$  whose projection  $I(e')$  contains  $v_i$  in its interior at a point  $p_{e'}$  whose projection  $I(p_{e'}) \in (I(v_i), I(v_i)+\epsilon)$ , where  $\epsilon$  is sufficiently small such that  $[I(v_i), I(v_i)+\epsilon] \times \mathbb{S}^1$  is free of edge crossings. Let  $G_{i+1}$  denote the resulting graph. Clearly, such subdivisions have no effect on the embeddability and do not alter the value  $N$ . The order  $\mathcal{O}'_i$  is obtained by taking  $\mathcal{D}_i \cap ((I(v_i) + \epsilon) \times \mathbb{S}^1)$ . There are two cases to distinguish depending on whether  $v_{i+1}$  is a source.

*The vertex  $v_{i+1}$  is not a source.* First, if  $v_{i+1}$  is not a source let  $e$  denote a lower edge at  $v_{i+1}$ , see Figure 2. In  $\mathcal{D}_i$  we clear  $e$  of crossings by using Lemma 4.

Now, if the lower edges at  $v_{i+1}$  do not appear consecutively in  $\mathcal{O}'_i$  we necessarily obtain a pair of edges crossing an odd number of times (contradiction). If  $e$  is

crossing free, every pair of edges cross in  $[I(v_i), I(v_{i+1})] \times \mathbb{S}^1$  an even number of times, if and only if  $\mathcal{D}_i \cap (I(v_{i+1}) \times \mathbb{S}^1)$  yields the same circular order as  $\mathcal{O}'_i$ . We continuously deform the drawing in  $(I(v_i) - \epsilon, I(v_{i+1}) + \epsilon) \times \mathbb{S}^1$  so that the order of the edges at  $I(v_{i+1}) \times \mathbb{S}^1$  is the same as in  $\mathcal{O}'_i$  while keeping  $e$  crossing free. Thus, in particular we do not perform any edge-vertex switch during the deformation. Now, every pair of edges in  $[I(v_i), I(v_{i+1})] \times \mathbb{S}^1$  cross an even number of times. Thus, we can replace the pieces of edges in  $[I(v_i), I(v_{i+1})] \times \mathbb{S}^1$  by geodesics connecting the same ends, which yields a desired  $\mathcal{D}_{i+1}$ . Note that the lower rotation at  $v_{i+1}$  is preserved since every pair of edges cross in  $[I(v_i), I(v_{i+1})] \times \mathbb{S}^1$  an even number of times.



**Fig. 2.** Case:  $v_{i+1}$  is not a source. (a) Drawing of  $[I(v_i), I(v_{i+1})] \times \mathbb{S}^1$  (left and right sides are identified), dashed curves may cross. (b) Edges crossing  $I(v_{i+1}) \times \mathbb{S}^1$  are deformed in a small neighborhood of  $I(v_{i+1}) \times \mathbb{S}^1$  so their cyclic order matches  $\mathcal{O}_i$ . (c) Edges are redrawn using geodesics.

*The vertex  $v_{i+1}$  is a source.* Let  $f$  denote the face in which  $v_{i+1}$  has the concave wedge. Let  $v$  denote the local maximum of  $f$  which is not the maximum of  $f$ . If

$v$  does not exist it follows that the maximum of  $f$  is a cut-vertex, and we let  $v$  be this cut-vertex. Let  $u$  denote the minimum of  $f$ . Let  $P$  denote the path in the walk  $f$  from  $u$  to  $v$  not containing  $v_{i+1}$ . Let  $Q$  denote the path in the walk  $f$  from  $v_{i+1}$  to  $v$  not containing  $u$ . In  $\mathcal{D}_i$  we clear  $P$  of crossings by a repeated application of Lemma 4.

Unlike the previous case, we cannot, in general, deform the edges in  $(I(v_i) - \epsilon, I(v_{i+1}) + \epsilon) \times \mathbb{S}^1$  while keeping  $P$  crossing free to obtain the order of edges at  $I(v_{i+1}) \times \mathbb{S}^1$  compatible with  $\mathcal{O}'_i$ . This might require an edge-vertex switch with  $v_{i+1}$  yielding a pair of edges crossing an odd number of times. Let  $S$  denote the cyclic interval between  $v_{i+1}$  and  $P$  in  $(I(v_{i+1}) \times \mathbb{S}^1)$  corresponding to the interior of  $f$  determined by the order of end pieces of  $P$  and  $Q$  in the upper rotation at  $v$ . The problem arises if the interior of  $S$  is crossed by edges. Thus, we first have to clean  $S$  of edges. We proceed by reducing  $G_{i+1}$  so that in the reduced instance no edge crosses  $S$ .

From now on, an edge  $e$  is always an edge such that  $I(e)$  contains  $I(v_{i+1})$  in its interior. Let  $G_e$  denote the connected component of  $G_{i+1}[\{v_{i+1}, \dots, v_n\}] \cup \{e\}$  containing  $e$ , where  $G_{i+1}[\{v_{i+1}, \dots, v_n\}]$  denotes the induced subgraph of  $G$  on  $\{v_{i+1}, \dots, v_n\}$ . An edge  $e$  *sprouts* beyond  $v$  if  $\max(I(G_e)) \geq I(v)$ .

The crucial observation is, that after we cleaned  $P$  of crossings,

- (i)  $S$  is crossed only by edges  $e$  not sprouting beyond  $v$ ; and
- (ii) the union  $G'_i$  of subgraphs  $G_e$ , for  $e$ 's not sprouting beyond  $v$ , does not contain an essential cycle. The smaller vertices of  $e$ 's are *seeds* of  $G'_i$ . (We recall that after the subdivisions  $I(G_e) \subseteq [I(v_i), 1]$ .)

Since the interior of  $(I(P) \times \mathbb{S}^1) \setminus P$  is homeomorphic to the plane, (ii) is obvious once we establish (i). To prove (i), consider a closed curve  $C$  obtained by concatenating the part of  $f$  between  $v_{i+1}$  and  $v$  (not containing  $u$ ); the part of  $P$  between  $v$  and  $I(v_{i+1}) \times \mathbb{S}^1$ ; and the part of  $I(v_{i+1}) \times \mathbb{S}^1$  crossing the interior of  $f$ , i.e., not crossing  $P$ . Claim (i) follows since  $e$  crosses  $C$  an odd number of times, and hence, the only way for  $e$  to sprout beyond  $v$  is to have an edge in  $G_e$  splitting a wedge in the interior of  $f$  which is impossible.

If  $G'_i$  is empty,  $S$  is not crossed by an edge, and hence, the order of edges in  $\mathcal{O}_{i+1}$  is inherited from  $\mathcal{O}'_i$ . The drawing  $\mathcal{D}_{i+1}$  is obtained analogously as in the case when  $v_{i+1}$  was not a source. Otherwise, consider a connected component  $H$  of  $G'_i$  given by the even radial drawing inherited from the drawing  $\mathcal{D}_i$  of  $G_{i+1}$ . Let  $S_H \subseteq \mathbb{S}^1$  denote the smallest connected subset such that the set  $\mathcal{C}_H = [I(v_i), I(v_i) + \epsilon] \times S_H$  contains all seeds of  $H$  and does not intersect  $P$ . Let  $H'$  denote the union of  $H$  with all  $G_e$  for  $e$  intersecting  $\mathcal{C}_H$ . By evenness of the drawing  $I(H') \subseteq [I(v_i), \max(I(H))]$ , since the curve  $(I(v_i) + \epsilon) \times S_H$  and a part of the outer face of  $H$  in  $[I(v_i) + \epsilon, 1] \times \mathbb{S}^1$  disjoint from  $P$  define a (non-essential) cycle having the non-seed vertices of  $V(H') \setminus V(H)$  in its interior. Let  $z$  be the vertex, for which  $I(z) = \max(I(H'))$ . Then  $z$  has a concave wedge inside a face  $f'$  for which  $\max(f') > \max(I(H'))$ . Indeed,  $f'$  is not an outer face by Lemma 1, since  $f'$  has a local maximum whose corresponding wedge is concave and different from  $v_n$ . Thus, the maximum of  $f'$  is different from  $z$  which is a local maximum of  $f'$ .

We reduce  $G_{i+1}$  by removing the edges of  $H'$  from it, and identifying the seeds of  $H'$  thereby replacing them by a single vertex  $s_i$ . In  $\mathcal{D}_i$ , this corresponds to contracting  $\mathcal{C}_H$  to a point while keeping the drawing radial and even. Let  $G'$  denote the resulting graph. Note that  $N$  necessarily decreased in  $G'$ . Indeed, at least the contribution of the source-sink pair  $(v_{i+1}, z)$  towards  $N$  decreased. Since  $G_{i+1}$  was a minimal counterexample,  $G'$  has a desired radial embedding. Similarly, let  $H''$  denote the graph obtained from  $H'$  by identifying its seeds and denoting the resulting vertex by  $s'_i$ . By the weak variant of the Hanani-Tutte theorem for monotone drawings, Theorem 3, we obtain a radial embedding of  $H''$  without an essential cycle. We combine the embeddings of  $H''$  and  $G'$  by identifying  $s_i$  and  $s'_i$ , and uncontracting  $s_i$ . We have room to accommodate  $H''$  inside  $f'$ , since  $\max(f') > \max(I(H'))$ . ■

## 4 Algorithm

Theorem 1 reduces the problem of radial planarity testing with a fixed rotation system to a system of linear equations over  $\mathbb{Z}_2$ . For planarity testing, systems like this were first constructed by Wu and Tutte [19, Section 1.4.2].

Unlike the  $x$ -monotone case, two drawings of an edge  $e$  with fixed endpoints may not be obtainable from each other by a continuous deformation of  $e$ , while keeping  $e$  radial: two radial drawings of  $e$  may also differ by a number of (Dehn) twists. The system has a variable  $x_{e,v}$  for every edge-vertex switch  $(e, v)$  such that  $I(v) \in I(e)$ , and  $x_e$  for every edge twist. Since we work in  $\mathbb{Z}_2$  orientation of a twist does not matter. We consider a twist of  $e = uv$ ,  $u < v$ , as being performed very close to  $v$ , i.e., the twist is carried out by removing a small portion  $P_e$  of  $e$  such that we have  $I(w) \notin I(P_e)$ , for all vertices  $w$ , and reconnecting the severed pieces of  $e$  by a curve intersecting every edge  $e'$ , s.t.  $I(P_e) \subset I(e')$ , exactly once. Observe that with respect to the parity of crossings between edges performing a twist close to  $v$  equals performing an edge-vertex switch of  $e$  with all the vertices  $w < v$  (even those  $w$  for which  $w < u$ ). Hence, any twist of  $e$  keeping  $e$  radial can be simulated by a twist of  $e$  very close to  $v$  and a set of edge-vertex switches of  $e$  with certain vertices  $w$ , for which  $u < w < v$ .

By the previous paragraph a linear system for testing radial planarity with the fixed rotation system looks as follows. Given an arbitrary radial drawing of  $G$  we denote by  $cr(e, f)$  the parity of the number of crossings between  $e$  and  $f$ . In the linear system, for each pair of independent edges  $(e, f) = (uv, wz)$ , where  $u < v$ ,  $w < z$ ,  $u < w$ , and  $w < v$ , we have  $x_{e,w} + x_{e,z} + x_f = cr(e, f)$  if  $z < v$ , and  $x_{e,w} + x_{f,v} + x_e = cr(e, f)$  if  $z > v$ . For a pair of edges  $(e, f) = (uv, uw)$ , where  $u < v < w$ , we have  $x_{f,v} + x_e = cr(e, f)$  and for a pair of edges  $(e, f) = (uv, uw)$ , where  $u > v > w$ , we have  $x_{f,v} + x_e + x_f = cr(e, f)$ . For a pair of edges  $(e, f) = (uv, uv)$ , where  $u < v$ , we have  $x_e + x_f = cr(e, f)$ .

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