



Hanani-Tutte for Radial Planarity

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Abstract

A drawing of a graph G is *radial* if the vertices of G are placed on concentric circles C_1, \dots, C_k with common center c , and edges are drawn *radially*: every edge intersects every circle centered at c at most once. G is *radial planar* if it has a radial embedding, that is, a crossing-free radial drawing. If the vertices of G are ordered or partitioned into ordered levels (as they are for leveled graphs), we require that the assignment of vertices to circles corresponds to the given ordering or leveling.

We show that a graph G is radial planar if G has a radial drawing in which every two edges cross an even number of times; the radial embedding has the same leveling as the radial drawing. In other words, we establish the weak variant of the Hanani-Tutte theorem for radial planarity. This generalizes a result by Pach and Tóth.

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1 Introduction

In a *leveled* graph every vertex is assigned a level in $\{1, \dots, k\}$. We can capture the leveling of the graph visually, by placing the vertices on parallel lines or concentric circles corresponding to the levels of G . To further emphasize the levels, we can require that edges respect the levels in the sense that edges must lie between the levels of their endpoints, and be monotone in the sense that they intersect any line (circle) parallel to (concentric with) the chosen lines (circles) at most once. If we choose lines, we obtain the concept of *level-planarity*; for circles we get *radial (level) planarity*.

Radial planarity was introduced by Bachmaier, Brandenburg and Forster [1] as a generalization of level-planarity [7]. Radial layouts are a popular visualization tool (see [8] for a recent survey); early examples of radial graph layouts can be found in the literature on sociometry [16]. Bachmaier, Brandenburg and Forster [1] showed that radial planarity can be tested, and an embedding can be found, in linear time. Their algorithm is based on a variant of PQ-trees [2] and is rather intricate. It generalizes an earlier linear-time algorithm for level-planarity testing by Jünger and Leipert [14]. In this paper, we take the first step toward an alternative algorithm for radial planarity testing via a Hanani-Tutte style characterization.

The classical Hanani-Tutte theorem [5, 25] states that a graph is planar if and only if it can be drawn in the plane so that every two independent edges cross an even number of times. A particularly nice algorithmic consequence of this result is that it reduces planarity testing to solving a system of linear equations (of polynomial size) over \mathbb{Z}_2 , a purely algebraic problem, which can be solved in polynomial time.

If we could show that a leveled graph G is radial planar if it has a radial drawing (respecting the leveling) in which every two *independent* edges cross an even number of times, we would have a new, simple, polynomial-time algorithm for radial planarity. We take the first step toward this result: a *weak* Hanani-Tutte theorem. A weak variant of the Hanani-Tutte theorem makes the stronger assumption that *every* two edges cross an even number of times. Often, this leads to stronger conclusions. For example, it is known that if a graph can be drawn in a surface so that every two edges cross evenly, then the graph has an embedding on that surface with the same rotation system, i.e. the cyclic order of the ends of edges at each vertex remains the same [3, 19].

Our main result, proved in Section 3, is the following theorem:

Theorem 1 *If a leveled graph has a radial drawing in which every two edges cross an even number of times, then it has a radial embedding with the same rotation system and leveling.*

Theorem 1 implies a polynomial-time algorithm for radial planarity testing of a leveled graph G if a combinatorial embedding (rotation system) of G is fixed. This algorithm (sketched in Section 4) is based on solving a system of linear equations over \mathbb{Z}_2 , see also [22, Section 1.4]. Thus, our algorithm runs

in time $O(|V(G)|^{2\omega})$, where $O(n^\omega)$ is the complexity of multiplication of two square $n \times n$ matrices. Since our linear system is sparse, it is also possible to use Wiedemann’s randomized algorithm [26], with expected running time $O(n^4 \log^2 n)$ in our case.

Remark 1 *In a sequel to this paper we prove the strong Hanani-Tutte theorem for radial planarity [10], using Theorem 1 as the base case (mirroring the development for level-planarity).*

Theorem 1 is a generalization of a weak variant of the Hanani-Tutte theorem for level-planarity¹, first proved by Pach and Tóth [17, 11], the result also follows from the proof of a more general result by M. Skopenkov [24]. The full Hanani-Tutte theorem for level-planarity was established only more recently [11], and it led to a quadratic time level-planarity test. A computational study of Chimani and Zeranski [4] of various algorithms for upward planarity testing (an NP-complete problem related to level-planarity), showed that the algorithm based on the Hanani-Tutte characterization of level-planarity performs very well in practice (it beats all other algorithms in nearly all scenarios).

Hanani-Tutte style characterizations have also been established for partially embedded planar graphs, several classes of simultaneously embedded planar graphs [23], and two-clustered graphs [9]. The family of counterexamples in [9, Section 6] (also, in a slightly different context, in [21, Example 1.6]) shows that a straightforward variant of the Hanani-Tutte theorem for clustered graphs with more than two clusters fails. Gutwenger et al. [13] showed that by using the reduction from [23], this counterexample can be turned into a counterexample for a variant of the Hanani-Tutte theorem for two simultaneously embedded planar graphs [23, Conjecture 6.20]. For higher-genus (compact) surfaces, the weak variant is known to hold in all surfaces [3, 20], while the strong variant is known for the projective plane only [18, 6]. It remains an intriguing open problem whether the strong Hanani-Tutte theorem holds for closed surfaces other than the sphere and projective plane.

2 Terminology

For the purposes of this paper, graphs may have multiple edges, but no loops. An *ordered* graph $G = (V, E)$ is a graph whose vertices are equipped with a total order $v_1 < v_2 < \dots < v_n$. We consider an ordered graph as a special case of a *leveled* graph, in which every vertex of G is assigned a *level*, a number in $\{1, \dots, k\}$ for some k . The leveling of the vertices induces a weak ordering of the vertices.

For convenience, we represent radial drawings as drawings on a (standing) cylinder. Intuitively, imagine placing a cylindrically-shaped mirror in the center

¹The result is stated for x -monotonicity, the special case of level-planarity in which every level contains a single vertex. As we will see below, this special case is equivalent to the general case.

of a radial drawing.² The *cylinder* is $I \times \mathbb{S}^1$, where I is the unit interval $[0, 1]$ and \mathbb{S}^1 is the unit circle. Thus, a *point* on the cylinder is a pair (i, s) , where $i \in I$ and $s \in \mathbb{S}^1$. The *projection* to I or \mathbb{S}^1 maps $(i, s) \in I \times \mathbb{S}^1$ to i , or s . We denote a projection of a point or a subset α of $I \times \mathbb{S}^1$ to I by $I(\alpha)$ and to \mathbb{S}^1 by $\mathbb{S}^1(\alpha)$. The *winding number* of a closed curve on a cylinder is the number of times the projection to \mathbb{S}^1 of the curve winds around \mathbb{S}^1 , i.e., the number of times the projection passes through an arbitrary point of \mathbb{S}^1 in the counterclockwise sense minus the number of times the projection passes through the point in the clockwise sense. A closed curve (or a cycle in a graph) on a cylinder is *essential* if it has an odd winding number.

A *radial drawing* of G is a drawing of G on the cylinder such that the projection to I of every edge is injective (i.e., an edge does not “turn back”) and for every pair of vertices $u < v$ we have $I(u) < I(v)$. We also speak of individual edges as being *radial* when they satisfy this condition. In a radial drawing an *upper (lower) edge* at v is an edge incident to v for which $\min I(e) = I(v)$ ($\max I(e) = I(v)$). A vertex v is a *sink (source)*, if v has no upper (lower) edges. In order to avoid unnecessary complications, we assume that $I(G)$ is contained in the interior of I .

The *rotation* at a vertex in a drawing (on any surface) of a graph is the cyclic, clockwise order of the ends of edges incident to the vertex in the drawing. The *rotation system* is the set of rotations at all the vertices in the drawing. In the case of radial drawings the *upper (lower) rotation* at a vertex v is the linear order of the end pieces of the upper (lower) edges in the rotation at v starting with the direction corresponding to the clockwise orientation of \mathbb{S}^1 . The rotation at a vertex in a radial drawing is completely determined by its upper and lower rotation. The *rotation system* of a radial drawing is the set of the upper and lower rotations at all the vertices in the drawing.

Since we work exclusively on a cylinder (or on a plane), we can *two-color* the complement of any closed curve (which may have self-crossings) so that connected regions each get one color and crossing the curve switches colors.³ See Figure 1. In a closed walk, it is possible for an edge to appear twice on that walk, in which case crossing the curve “switches colors twice”, meaning that the color is the same locally on both sides of any such edge.

When speaking of two edges in a drawing crossing *evenly*, or *oddly*, we are referring to the parity of the number of crossings between the two edges. A drawing of G is *even* if every two edges in the drawing cross an even number of times.

For any (non-degenerate) continuous deformation of a drawing of G , the parity of the number of crossings between a pair of edges changes only when an edge passes through a vertex during the deformation. We call this event an *edge-vertex switch*. Note that when an edge e passes through a vertex v the parity of the number of crossings between e and every edge incident to v changes.

²Search for “cylindrical mirror anamorphoses” on the web for many cool pictures of this transformation.

³This can be shown by starting with a two-coloring of the complement of a circle, and proving that the two-coloring can be maintained as the curve is gradually deformed.

3 Weak Hanani-Tutte for Radial Drawings

In this section, we prove Theorem 1, the weak Hanani-Tutte theorem for radial planarity. We claim that it will be sufficient to restrict ourselves to the special case in which every level of G contains a single vertex, an *ordered graph*.

Theorem 2 *If an ordered graph G has an even radial drawing, then it is radial planar. In this case it has a radial embedding with the same ordering and the same rotation system as the original drawing, and the parity of the winding number of every cycle remains the same.*

The reduction of Theorem 1 to Theorem 2 is based on the same construction used in [11, Section 4.2] to reduce level-planarity to x -monotonicity: Suppose we are given an even radial drawing of a leveled graph G . If any level of G contains more than one vertex, we do the following: if any vertex at that level is a source or a sink, we add a crossing-free edge on the empty side of that vertex. We place the new vertex at a new level, close to the current level we are working on. We now slightly perturb all the vertices of the current level so no two vertices are at the same level (without moving them past any of the new vertices we created). We can do so, while keeping all edges radial, and without introducing any crossings. Since the new vertices we added are at unique levels, we only perform the perturbation on the original levels, see Figure 2 for an illustration. Call the resulting ordered graph G' . By Theorem 2, G' has a radial embedding in which the rotation system, the levels of the vertices, and the parity of the winding numbers of cycles are the same as in the radial drawing of G' . We can now move all perturbed vertices back to their original levels, the additional edges we added ensure that this is always possible.

We will make use of the weak Hanani-Tutte theorem for x -monotone graphs due to Pach and Tóth [17], reproved in [11].

Theorem 3 (Pach, Tóth [17]) *Suppose that G can be drawn so that edges are x -monotone and every two edges cross an even number of times. Then there exists an embedding of G , in which the vertices are drawn as in the given drawing of G , the edges are x -monotone, and the rotation system is the same.*

Figure 3 shows an example of an ordered graph for which x -monotonicity and radial planarity differ. A radial embedding of a graph not admitting an x -monotone embedding, must contain an essential cycle (we will prove this later, in Lemma 7).

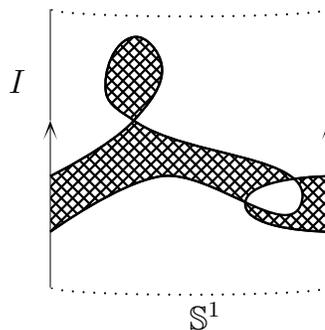


Figure 1: Two-coloring the complement of a closed curve on the cylinder. The left and right side are identified.

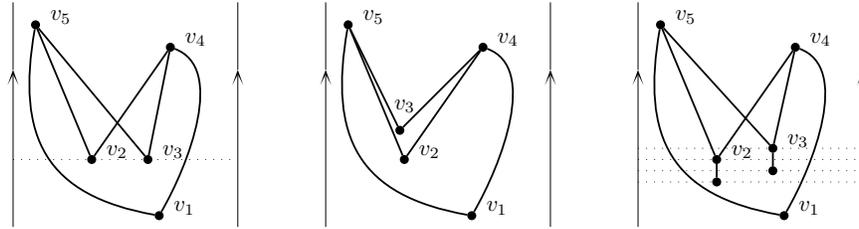


Figure 2: The leveled graph on the left does not admit a radial embedding with the same rotation system, v_2 and v_3 are sources at the same level. The middle illustration shows that perturbing vertices at the same level can turn a level graph that does not admit a radial embedding with the given rotation system into an ordered graph admitting such an embedding. The right illustration shows the construction: adding a crossing-free edge below both sources. The resulting ordered graph does not admit a radial embedding with the given rotation system.

3.1 Working with Even Radial Drawings

Given a connected graph G with a rotation system, we can define a *facial walk* purely combinatorially by following the edges according to the rotation system (see, for example, [12, Section 3.2.6]), where we traverse consecutive edges at each vertex in clockwise order.⁴ A vertex can occur multiple times on a facial walk (in which case it is a cut-vertex). Any drawing of a graph G on an orientable surface defines a rotation system. For an even drawing of a connected graph G on the plane, the rotation system describes an embedding of the graph in the plane, so that the facial walks correspond to actual faces. This is the essence of the weak Hanani-Tutte theorem in the plane:

Theorem 4 (Cairns, Nikolayevsky [3]⁵) *If a graph has an even drawing in the plane, then it has an embedding in the plane with the same rotation system.*

We need some terminology for radial drawings of an ordered graph G with $v_1 < v_2 < \dots < v_n$. The *maximum* (*minimum*) of a facial walk W in a radial drawing of G is the maximum (minimum) v so that v lies on W . A *local maximum* (*local minimum*) of a facial walk W is a vertex v on W so that $v > u$ and $v > w$ ($v < u$ and $v < w$), where u and w are the predecessor and successor of v on the facial walk W .

Let $e = uv$ and $e' = vw$ be two consecutive edges on a facial walk in a drawing of G . We call (e, v, e') a *wedge* at v , and we can identify it with a small neighborhood of v in the drawing enclosed by the ends of e and e' (in clockwise order) at v . Intuitively, we think of the wedges as being the corners of some face; Theorem 4 tells us that this intuition is justified for planar even drawings.

⁴Facial walks bound faces of an embedding of the graph in some surface.

⁵Cairns and Nikolayevsky prove this result for arbitrary orientable surfaces; a simple, direct proof for the plane can be found in Pelsmajer, Schaefer, Štefankovič [19].

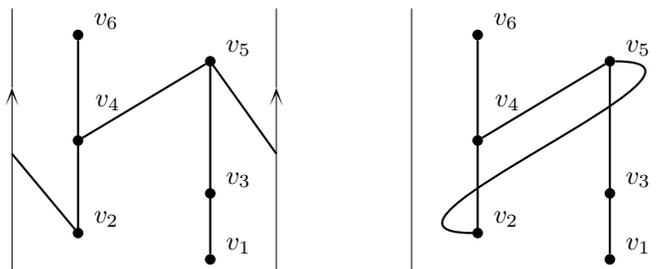


Figure 3: An instance of an ordered graph that admits a radial embedding (shown on the left) but not an x -monotone embedding with the same rotation system (an x -monotone drawing is shown on the right). The left and right sides of the left illustration are identified (as indicated by the arrows).

A point in the complement of W is in the *interior* (*exterior*) of W if it receives the same (opposite) color as a wedge in W when we two-color W . Note that in an even drawing all the wedges of a facial walk have the same color.

At a sink (source) v , the wedge that contains the region directly above (below) v is called a *concave wedge*. See Figure 4. A facial walk W in a radial drawing is an *upper* (*lower*) *facial walk* if its maximum (minimum) vertex is a sink (source) and the concave wedge at that vertex is part of the interior of W . An *outer facial walk* is an upper or lower facial walk; other facial walks are *inner facial walks*.

Lemma 1 *In an even radial drawing of a connected graph G , at most one facial walk has a concave wedge at its maximum (minimum) and it can only happen at v_n (v_1). Hence, G has either two outer facial walks (one lower and one upper) or exactly one outer facial walk (which is both lower and upper).*

Proof: Without loss of generality, suppose that G has a facial walk W with a concave wedge at its maximum u . Since u is the maximum of W and the wedge at u is concave, everything above u lies in the interior of W (as defined above). Hence, if there is a vertex $v > u$, such a vertex v lies in the interior of W . Since G is connected, there is a (shortest) path from v to W . Since all edges are even, this path (edges and vertices) lie in the interior of W , and thus the path must attach to W in its interior, contradicting the definition of facial walk. So $u = v_n$.

On the other hand, both v_1 and v_n are incident to concave wedges, so there always is a lower and an upper facial walk, though it is possible that they are the same. \square

Let C be a cycle in a radial drawing of G . Then C is non-essential if and only if in the two-coloring of (the complement of) C the concave wedge of C at the minimum and maximum of C receive the same color: if we draw a curve between those two wedges, the number of times it crosses C will have the same

parity as the winding number of C . For example, Figure 1 shows a non-essential curve, whereas $v_2v_4v_5$ in Figure 3 is essential.

Lemma 2 *The parity of the winding number of a cycle in an even radial drawing of a graph is determined by the rotation system of the graph.*

Proof: Let e, f and e', f' , respectively, be edges incident to the maximum v and minimum u of C . Let $<_v$ be the lower rotation at v and let $<_u$ be the upper rotation at u . Suppose that $e <_v f$ and suppose that e, e', f', f appear in this order along C . We will show that the cycle C winds evenly if $e' <_u f'$ and oddly if $f' <_u e'$.

Two-color the complement of C . Traverse the path in C which begins with v, e and ends with e', u . At the beginning, the colored region to the right includes the concave wedge at v . Since C is an even drawing, the color immediately to the right will be the same as we begin and end our path traversal. At the end, the colored region to the right includes the concave wedge at u if and only if $e' <_u f'$. As noted earlier, the concave wedges have the same color if and only if the winding parity is even. \square

In a radial drawing of G a vertex is *below* (*above*) an essential cycle C if it is not on C and has the same color as $0 \times \mathbb{S}^1$ ($1 \times \mathbb{S}^1$) in the two-coloring of the complement of C .

Lemma 3 *If an even radial drawing of a connected graph contains an essential cycle, then the graph has two outer facial walks.*

Proof: Let C be an essential cycle of a graph G in an even radial drawing. G can be split into two induced subgraphs G_L and G_U , where G_L is induced by the vertices on and below C and G_U is induced by the vertices on and above C . Note that by definition C (traversed counterclockwise) is the upper outer facial walk of G_L and (traversed clockwise) the lower outer facial walk of G_U . By Lemma 1, G_L has a lower facial walk W_L , and G_U has an upper facial walk W_U . Then W_L and W_U are also the lower and upper facial walks of G . If $W_L \neq W_U$, we are done. Otherwise, both walks are traversals of C (and $C = G$). Since C is essential, the concave wedges at its extrema have different colors in a 2-coloring of the complement of C , so the upper facial walk and the lower facial walk of C must be different. \square

Intuitively, an essential cycle separates the rest of the graph into the part above, and the part below it; the following lemma makes this precise.

Lemma 4 *Let P be a path and let C be an essential cycle, vertex disjoint from P , in an even radial drawing of a graph. Then $I(P)$ does not contain $I(C)$.*

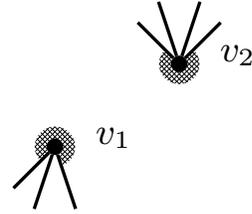


Figure 4: Two concave wedges, a sink, v_1 , on the left, and a source, v_2 , on the right.

Proof: Suppose $I(P)$ contains $I(C)$. We can then find a vertex u on P above C and a vertex v on P below C . Thus, the sub-path of P between u and v , and hence, an edge of P on the sub-path between u and v intersects an edge of C an odd number of times, which is a contradiction. \square

The next lemma allows us to simplify facial walks.

Lemma 5 *If an ordered graph has an even radial drawing, then we can augment the drawing of the graph by adding edges so that the resulting drawing is still even and radial, every inner facial walk has at most two local minima and two maxima, and each outer facial walk of each component has exactly one local minimum and one local maximum.*

For the proof of Lemma 5, we need the following lemma, which is a simple extension of a similar redrawing result for x -monotone drawings from [11]. Call an edge *bounded* if its points lie between its endpoints; that is, $I(u) < I(p) < I(v)$ for every point p in the interior of edge uv . Call the drawing of G *bounded* if all edges are bounded.

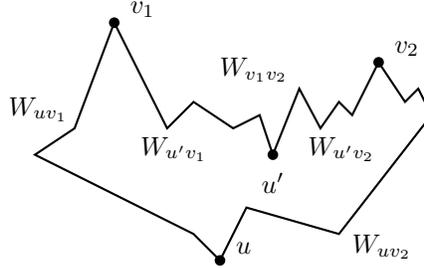
Lemma 6 *If an ordered graph has an even bounded drawing, then it has an even radial drawing with the same rotation system.*

Proof: It is sufficient to show how to redraw any particular edge $e = uv$ radially without changing the remainder of the drawing, and so that e remains even: While keeping $I(e) = [I(u), I(v)]$ and the rotation system fixed, we continuously deform e so that its projection to I becomes injective. As e is deformed, it will pass through some vertices an odd number of times; call this set of vertices S . To reestablish the original crossing parities between e and all other edges, we need to perform (e, w) -switches for every vertex $w \in S$. We can do so by deforming e inside $[I(w) - \epsilon, I(w) + \epsilon] \times \mathbb{S}^1$, ensuring that any new crossings with e (other than the ones due to pushing e over w) will come in pairs (keeping the drawing even) and so that e remains radial. \square

Proof of Lemma 5: We first describe how to reduce the number of local maxima in an inner facial walk if that number is at least three.

So, suppose there is an inner facial walk W which has at least three local maxima. Let u be the vertex of W with minimum I -coordinate, and let v_1, v_2 be two local maxima on W with largest I -coordinates. (It's possible that $v_1 = v_2$ since a vertex can appear more than once in W .) Then W is comprised of a u, v_1 -walk W_{uv_1} , a u, v_2 -walk W_{uv_2} , and a v_1, v_2 -walk $W_{v_1v_2}$. Letting u' be the vertex of minimum I -coordinate on $W_{v_1v_2}$, that walk is the union of a u', v_1 -walk $W_{u'v_1}$ and a u', v_2 -walk $W_{u'v_2}$. (As sub-walks of W , $W_{u'v_2}$ would be reversed.) Figure 5 illustrates the decomposition of W .

Note that for each $W' \in \{W_{uv_1}, W_{uv_2}, W_{u'v_1}, W_{u'v_2}\}$, the I -coordinates of the endpoints of W' are the maximum and minimum of $I(W')$. Since we assumed that there are at least three local maxima, at least one of these walks must contain a local maximum not at its endpoints; let W' be that walk. We add to G

Figure 5: Decomposed facial walk W .

an edge connecting the two endpoints of W' by following W' on its interior side (as determined by W), and then using Lemma 6 to make the resulting drawing even and radial. Then W is replaced by two facial walks, each containing the new edge; one of the new walks omits the internal local maximum of W' , and the other at least one of v_1, v_2 , so both of the new facial walks have fewer local maxima than W . By repeating this process we will eventually obtain a drawing in which every inner facial walk has at most two local maxima. The same procedure can be used for local minima (by reversing the underlying ordering of the graph). We conclude that every inner facial walk can be assumed to have at most two local minima and two local maxima.

Let W be an outer facial walk, and let u, v be its minimum/maximum vertex. If either (or both) of the u, v -subwalks of W has a local maximum or minimum not at its endpoints, use Lemma 6 to add an edge from u to v along the subwalk(s), starting and ending in the exterior of W . By Lemma 6 we can make the drawing even and radial. Hence, we end up with a new outer facial walk having exactly one minimum and maximum. \square

One consequence of Lemma 5 is the following useful fact. It allows us to treat non-essential components of a radial embedding as if they were edges (since x -monotone embeddings can be made arbitrarily narrow).

Lemma 7 *If a graph has an even radial drawing that contains no essential cycles, then it has an x -monotone embedding with the same rotation system.*

Proof: It is sufficient to prove this for the case that the graph is connected (since it is easy to combine x -monotone embeddings of different components). Using Lemma 5, we can assume that the outer facial walk consists of two edges between the minimum and maximum vertices of the graph (note that there is only one outer facial walk; if there were two, both would be essential). Using Lemma 8 from the next section, we can clean one of these edges of crossings. Since the edge is part of an outer facial walk (and the rotation system did not change), cutting the cylinder close to the crossing-free edge leaves us with an even x -monotone drawing of G in the plane. Theorem 3 now implies that the

graph has an x -monotone embedding with the same rotation system (we remove any edge we added during the process at this point). \square

3.2 Removing Radial Crossings

In this section we complete the proof of Theorem 2. We will make use of a simple redrawing tool that allows us to clear crossings with a single edge. This is a slight extension of redrawing results we have used in previous papers.

Lemma 8 *Suppose we are given a radial drawing of G (not necessarily even), and an even edge e of G . Then the edges crossing e can be redrawn inside $I(e) \times \mathbb{S}^1$ to make e crossing-free and keeping the drawing radial. The redrawing does not change the rotation system, the vertex locations, the crossing parity between any pair of edges, or the winding number parity of any cycles.*

We note that if the original drawing is even, then so is the redrawing.

Proof: Consider an edge f crossing e . Cut f wherever it crosses e . This leaves us with an even number of ends of f on each side of e (since f crossed e evenly). On each side of e pair up the ends in order, and reconnect them locally. Edge f may now consist of multiple components, but we can connect these by narrow tunnels to each other without changing the crossing parity between f and any other edge. However, f may no longer be radial. Since we can perform the reconnections so that the curve representing f lies strictly between its two endpoints, we can make f radial by using the same redrawing as in Lemma 6. Since e does not separate the cylinder, we can do so without crossing e . Repeating this for every edge f crossing e , removes all crossings with e . The redrawing of the edges does not affect the rotation system, the vertex locations, or the parity of the winding number of cycles. \square

In a first step, we show that we can assume that a counterexample to Theorem 2 has to be connected. Call a connected component of a radially drawn graph G *essential* if it contains an essential cycle.

Lemma 9 *A counterexample to Theorem 2 with the smallest number of vertices is connected.*

The following argument also shows that a minimal counterexample to the *strong* Hanani-Tutte theorem for radial planarity is connected.

Proof: For the sake of contradiction let G denote a minimal counterexample. We apply Lemma 5 to ensure that facial walks satisfy the restrictions established there.

Suppose that G contains a non-essential component G_0 . Apply induction to obtain a radial embedding of G_0 with the winding number parity of cycles preserved. Thus, the embedding has no essential cycles, so Lemma 7 implies that the embedding is x -monotone. Let $m_0 := \min I(G_0)$ and $M_0 := \max I(G_0)$. By Lemma 5, the outer face of G_0 is bounded by two radially-drawn paths from

$I = m_0$ to $I = M_0$. Then G_0 can be deformed to a new radial embedding so that G_0 lies in small neighborhood of any given radially-drawn curve \mathcal{C} with $I(\mathcal{C}) = [m_0, M_0]$. We can do likewise for every non-essential component. If every component is non-essential, it is easy to combine them to get an embedding of G , which satisfies Theorem 2. We conclude that G contains at least one essential component.

Let G_* be the union of the essential components of G . Apply induction to get a radial embedding of G_* . Let E' be the set of edges e in G_* with $M_0 \in I(e)$. Then $G_* - E'$ is the disjoint union of the two subgraphs G_*^-, G_*^+ induced by vertex sets $\{v \in V(G_*) : I(v) < M_0\}$ and $\{v \in V(G_*) : I(v) > M_0\}$, respectively.

Suppose that the upper outer face of G_*^- lies entirely above $I = m_0$. Then its boundary includes an essential facial walk W_U , for which $m_0 < I(W_U) < M_0$. Then $I(G_0)$ contains $I(W_U)$, contradicting Lemma 4.

Thus, the upper outer face of G_*^- intersects $I = m_0$. Every face of G_* in the upper outer face of G_*^- intersects $I = M_0$, so G_* has a face f that intersects both $I = m_0$ and $I = M_0$. By Lemma 5, the boundary of f includes a radially-drawn path between its minimum and maximum vertices. An embedding of G_0 can be inserted into f alongside that path. We can do likewise for every non-essential component of G , so that none of them intersect each other. Thus, we may assume that all components of G are essential.

Let G_1 be the component containing the minimum vertex v_1 , and let $G_2 = G \setminus G_1$. Let W_1 be the upper facial walk of G_1 and let W_2 be the lower facial walk of G_2 . Since W_2 is essential and $\min I(W_2) > I(v_1)$, Lemma 4 implies that $\max I(W_2) > \max I(W_1)$. Then since W_1 is essential, $\min I(W_2) > \min I(W_1)$ by Lemma 4.

By Lemma 5, W_1 consists of two radially-drawn curves. We can deform the radial embedding of G_1 within the region $\min W_1 < I < \max W_1$ so that, except for a small strip near $I = \min W_1$, G_1 occupies only the a small neighborhood of a single radial curve (see Figure 6). Likewise we deform the radial embedding of G_2 so that on the region $\min W_2 < I < \max W_2$ it lies very near a radial curve, except $I = \min W_2$. Then it is easy to combine the two resulting radial embeddings of G_1 and G_2 to get an embedding of G , which satisfies Theorem 2. \square

Proof of Theorem 2: For a contradiction, suppose there is a counterexample to the theorem. We can assume that it has no inner facial walks with exactly two edges. If not, remove one such edge from a counterexample with the fewest number of such walks. We can embed it by induction, then add the edge back to the embedding to satisfy Theorem 2.

Among such counterexamples, choose G to be one with the fewest number of vertices, then the largest number of edges. By Lemma 9 we know that G is connected, and Lemma 5 tells us that every inner facial walk of G contains at most two local minima and maxima, and every outer facial walk at most one local minimum and maximum.

We will prove that for $1 \leq i \leq n$, G has an even radial drawing D_i with the given rotation system, so that the restriction of D_i to $[I(v_1), I(v_i)] \times \mathbb{S}^1$ is

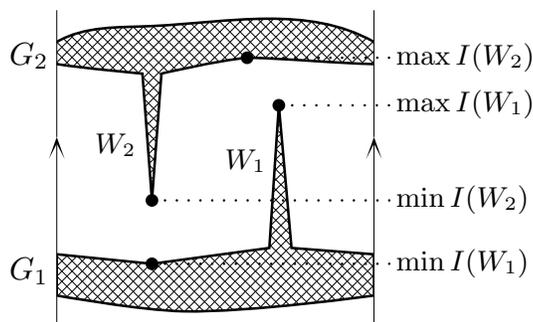


Figure 6: The radial embedding of G_1 , deformed so that except for a small strip near $I = \min W_1$, it occupies only a “skinny” region. Likewise for G_2 , so such embeddings are easily combined with no overlap.

crossing-free. Then D_n restricted to G will be the desired drawing, completing the proof.

We may let D_1 be the given drawing of G . Suppose that we have the drawing D_i of G_i as described above for some $i < n$. We will show how to obtain D_{i+1} . We distinguish two cases based on whether v_{i+1} is a source or not.

Vertex v_{i+1} is not a source. Let e be any lower edge at v_{i+1} . Use Lemma 8 to clear e of crossings, without affecting the drawing within the region $[I(v_1), I(v_i)] \times \mathbb{S}^1$, leaving it crossing-free as before. See Figure 7(a).

Let π_i be the cyclic order that elements of G intersect the circle $I(v_i) \times \mathbb{S}^1$, except replace its only vertex v_i by the edges in the upper rotation of v_i (in that order).

The edges in the lower rotation at v_{i+1} must be consecutive in π_i : if not, then π_i has a subsequence a, b, c, d where a, c are incident to v_{i+1} and b, d are not. But then b or d must cross a or c an odd number of times—a contradiction.

We can change the cyclic order in which edges intersect $I(v_{i+1}) \times \mathbb{S}^1$ to any other cyclic order while maintaining a radial drawing, by continuously deforming edges within the region $([I(v_{i+1}) - \epsilon, I(v_{i+1}) + \epsilon] \times \mathbb{S}^1) \setminus (e \cup v_{i+1})$. See Figure 7(b). The modified drawing will still be even since that region contains no vertices. We will make the order in which edges intersect $I(v_{i+1}) \times \mathbb{S}^1 \setminus v_{i+1}$ match the order obtained from π_i after removing the (consecutive) edges which are incident to v_{i+1} . Then every pair of edges crosses an even number of times within $[I(v_i), I(v_{i+1})] \times \mathbb{S}^1 \setminus e$. Finally, edges can easily be redrawn in that region (as geodesics) with no crossings in that region, giving us D_{i+1} as desired. See Figure 7(c).

Vertex v_{i+1} is a source. Let W be the facial walk in which v_{i+1} occurs in the concave wedge. Since v_{i+1} is a source, W must have two local maxima (possibly the same vertex). Let z be the maximum vertex of W and let v be the other local maximum of W . Let u denote the minimum of W . By Lemma 5,

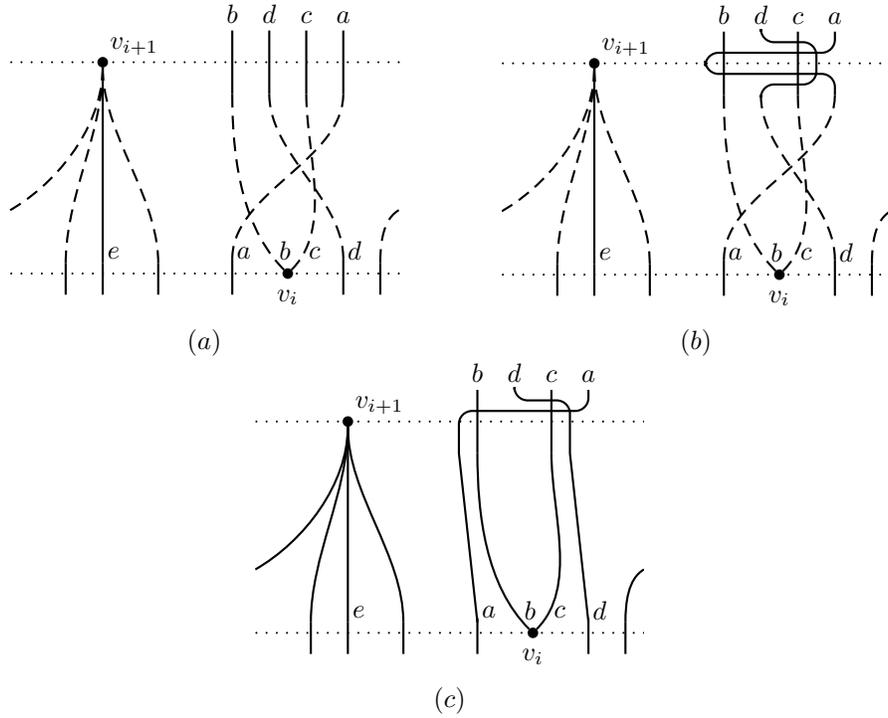


Figure 7: Case: v_{i+1} is not a source. (a) Drawing of $[I(v_i), I(v_{i+1})] \times \mathbb{S}^1$ (left and right sides are identified), dashed curves may cross. (b) Edges crossing $I(v_{i+1}) \times \mathbb{S}^1$ are deformed in a small neighborhood of $I(v_{i+1}) \times \mathbb{S}^1$ so their cyclic order matches \mathcal{O}_i . (c) Edges are redrawn using geodesics.

W is comprised of four radially-drawn paths: a u, v -path P , a v, v_{i+1} -path Q , a v_{i+1}, z -path R and a z, u -path. In D_i we clear P of crossings by repeatedly applying Lemma 8 to each edge of P .

The circle $I(v_{i+1}) \times \mathbb{S}^1$ is broken into two curves by v_{i+1} and P ; let S be the one which reaches P from the interior of W . Similarly, the circle $I(v) \times \mathbb{S}^1$ is split into two curves by v and R ; let T be the one which reaches R from the exterior of W . (See Figure 8.) If S is free of crossings, then there is room alongside P to draw a crossing-free edge from v_{i+1} to u ; we can then apply induction to finish the proof.

Our goal is to clear S of crossings while keeping the drawing even, and P crossing-free. Consider the simple closed curve formed by S, Q , and the part of P that connects v to S ; let V_S be the set of vertices in its interior (note that v and v_{i+1} lie in its exterior). Consider the simple closed curve formed by T, Q , and the part of R that connects v_{i+1} to T ; let V_T contain the vertices on that curve and in its interior (the side that does not contain V_S). Then V_S and V_T

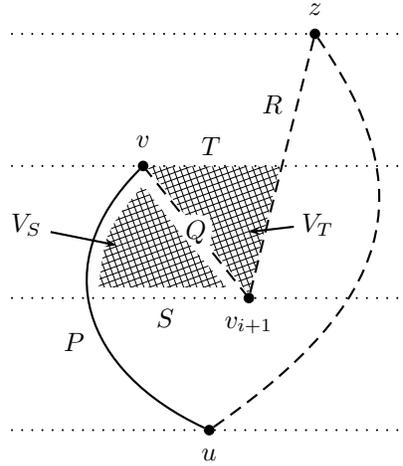


Figure 8: W , for the case that v_{i+1} is a source. Dashed curves may cross. Dotted curves are not part of G itself. Each shaded region is defined by a two-coloring of the complement of a closed curve.

are disjoint.

Let E_S be the set of edges incident to vertices in V_S . Any edge from V_S to its complement must cross S oddly (exactly once), since P is crossing-free and edges of Q are even; hence the other endpoint lies below $I = I(S)$.

Let E_T be the set of edges incident to vertices in V_T . Consider any edge e from V_T to its complement. Since W is a face, e reaches V_T from the interior of T . Then e crosses T oddly (exactly once) or e crosses R oddly below $I = I(T)$ in which case e crosses R oddly above $I = I(T)$, too. In either case, the other endpoint of e lies above $I = I(T)$. Thus, E_S and E_T are disjoint and share no endpoints.

For every vertex w and edge e with $I(w) \in I(e)$, perform a radial (e, w) -move without crossing P if either (i) $e \in E_S$ and $w \in V_T$, or (ii) $e \in E_T$ and $w \in V_S$. We claim that the resulting drawing is still even. Indeed, only a pair of edges $e_S \in E_S, e_T \in E_T$ may change crossing parity—once for each endpoint in $I(e_S) \cap I(e_T)$ which is in $V_S \cup V_T$. If $I(e_S) \cap I(e_T) \neq \emptyset$, then it contains exactly two endpoints of $\{e_S, e_T\}$. If an endpoint of e_S or e_T is not in $V_S \cup V_T$, then it lies in the region $I > I(T)$ or $I < I(S)$, so it is not in $I(e_S) \cap I(e_T)$. Thus, the crossing parity of e_S, e_T remains even.

The (e, w) -moves with $w = v_{i+1}$ moved every crossing with S past v_{i+1} and off of S . No other (e, w) -move affects the region near $I = I(v_{i+1})$, so S is now free of crossings. So, we can continue as before.

Finally, the invariance of the parity of winding numbers follows from Lemma 2 and the fact that the rotation system of the embedding is the same as that of the original drawing. \square

4 Algorithm

Theorem 1 allows us to reduce the algorithmic problem of radial planarity testing with a fixed rotation system to a system of linear equations over $\mathbb{Z}/2\mathbb{Z}$. For planarity testing, systems like this were first constructed by Wu and Tutte [22, Section 1.4.2].

Unlike in the case of x -monotone drawings, two drawings of an edge e with end vertices fixed cannot necessarily be obtained one from another by a continuous deformation during which we keep the drawing of e radial: up to a continuous deformation, two radial drawings of an edge differ by a certain number of (Dehn) twists. We perform a twist of $e = uv$, $u < v$ very close to v , i.e., the twist is carried out by removing a small portion P_e of e such that we have $I(w) \notin I(P_e)$, for all vertices w , and reconnecting the severed pieces of e by a curve intersecting every edge e' , s.t. $I(P_e) \subset I(e')$, exactly once. Observe that with respect to the parity of crossings between edges performing a twist close to v equals performing an edge-vertex switch of e with all the vertices $w < v$ (even those w for which $w < u$). Hence, the orientation of the twist does not matter, and any twist of e keeping e radial can be simulated by a twist of e very close to v and a set of edge-vertex switches of e with certain vertices w , for which $u < w < v$.

By the previous paragraph a linear system for testing radial planarity with the fixed rotation system can be constructed as follows. The system has a variable $x_{e,v}$ for every edge-vertex switch (e, v) such that $I(v) \in I(e)$, and a variable x_e for every edge twist. Given an arbitrary radial drawing of G we denote by $\text{cr}(e, f)$ the parity of the number of crossings between e and f . In the linear system, for each pair of independent edges $(e, f) = (uv, wz)$, where $u < v$, $w < z$, $u < w$, and $w < v$, we require

$$\text{cr}(e, f) \equiv \begin{cases} x_{e,w} + x_{e,z} + x_f \pmod{2} & \text{if } z < v, \text{ and} \\ x_{e,w} + x_{f,v} + x_e \pmod{2} & \text{if } z > v. \end{cases}$$

For a pair of dependent edges $(e, f) = (uv, uw)$ we require that

$$\text{cr}(e, f) \equiv \begin{cases} x_{f,v} + x_e \pmod{2} & \text{if } u < v < w, \text{ and} \\ x_{f,v} + x_e + x_f \pmod{2} & \text{if } u > v > w, \text{ and} \\ x_e + x_f \pmod{2} & \text{if } u < v = w. \end{cases}$$

This (sparse) linear system over $\text{GF } 2$ is solvable if and only if G has a radial embedding with the given rotation system.

5 Open Questions

In a second part of this paper, we establish the strong Hanani-Tutte theorem for radial drawings, answering the main open question left by the current paper, but there are other follow-up questions specifically linked to the weak variant. The weak Hanani-Tutte theorem can be strengthened in various ways; for example, Loeb and Masbaum showed that if a graph can be drawn in the plane so that

every even subgraph (a subgraph is *even* if all its vertices have even degree) has an even number of self-crossings, then the graph is planar, and can be embedded with the same embedding scheme [15]. Does a similar result hold for x -monotone or radial drawings? We have to assume that the graph is 2-connected (since trees can fail to be x -monotone for specific levelings). For the plane, it is known that if every cycle has an even number of self-crossing, then the graph is planar (though the rotation may have to change) [22]. Again, it is open whether a similar result holds for x -monotone or radial drawings.

By collapsing both $0 \times \mathbb{S}^1$ and $1 \times \mathbb{S}^1$ to a point, we can treat radial drawings as a special case of graph drawings on the piece-wise linearly embedded sphere \mathcal{S} in Euclidean three-space, where levels are given by a single coordinate, call it z . Does our result extend to drawings of this type? Does it make a difference if for every vertex v on level i we also prescribe a connected component of $\{(x, y, z) \in \mathbb{R}^3 \mid z = i\} \cap \mathcal{S}$ to which v belongs in a drawing?

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