Decomposing Permutation Automata

Ismaël Jecker
Institute of Science and Technology, Klosterneuburg, Austria

Nicolas Mazzocchi
IMDEA Software Institute, Madrid, Spain

Petra Wolf
Fachbereich IV, Informatikwissenschaften, Universität Trier, Germany

Abstract

A deterministic finite automaton (DFA) \( A \) is composite if its language \( L(A) \) can be decomposed into an intersection \( \bigcap_{i=1}^{k} L(A_i) \) of languages of smaller DFAs. Otherwise, \( A \) is prime. This notion of primality was introduced by Kupferman and Mosheiff in 2013, and while they proved that we can decide whether a DFA is composite, the precise complexity of this problem is still open, with a doubly-exponential gap between the upper and lower bounds. In this work, we focus on permutation DFAs, i.e., those for which the transition monoid is a group. We provide an NP algorithm to decide whether a permutation DFA is composite, and show that the difficulty of this problem comes from the number of non-accepting states of the instance: we give a fixed-parameter tractable algorithm with the number of rejecting states as the parameter. Moreover, we investigate the class of commutative permutation DFAs. Their structural properties allow us to decide compositionality in NL, and even in LOGSPACE if the alphabet size is fixed. Despite this low complexity, we show that complex behaviors still arise in this class: we provide a family of composite DFAs each requiring polynomially many factors with respect to its size. We also consider the variant of the problem that asks whether a DFA is \( k \)-factor composite, that is, decomposable into \( k \) smaller DFAs, for some given integer \( k \in \mathbb{N} \). We show that, for commutative permutation DFAs, restricting the number of factors makes the decision computationally harder, and yields a problem with tight bounds: it is \( \text{NP} \)-complete. Finally, we show that in general, this problem is in \( \text{PSPACE} \), and it is in \( \text{LOGSPACE} \) for DFAs with a singleton alphabet.

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1 Introduction

Compositionality is a fundamental notion in numerous fields of computer science [3]. This principle can be summarised as follows: Every system should be designed by composing simple parts such that the meaning of the system can be deduced from the meaning of its parts, and how they are combined. For instance, this is a crucial aspect of modern software engineering: a program split into simple modules will be quicker to compile and easier to maintain. The use of compositionality is also essential in theoretical computer
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Science: it is used to avoid the state explosion issues that usually happen when combining parallel processes together, and also to overcome the scalability issues of problems with a high theoretical complexity. In this work, we study compositionality in the setting of formal languages: we show how to make languages simpler by decomposing them into intersections of smaller languages. This is motivated by the model-checking problems. For instance, the LTL model-checking problem asks, given a linear temporal logic formula \( \varphi \) and a finite state machine \( M \), whether every execution of \( M \) satisfies \( \varphi \). This problem is decidable, but has a high theoretical complexity (PSPACE) with respect to the size of \( \varphi \) [1]. If \( \varphi \) is too long, it cannot be checked efficiently. This is where compositionality comes into play: if we can decompose the specification language into an intersection of simple languages, that is, decompose \( \varphi \) into a conjunction \( \varphi = \varphi_1 \land \varphi_2 \land \cdots \land \varphi_k \) of small specifications, it is sufficient to check whether all the \( \varphi_i \) are satisfied separately.

Our aim is to develop the theoretical foundations of the compositionality principle for formal languages by investigating how to decompose into simpler parts one of the most basic models of abstract machines: deterministic finite automata (DFAs). We say that a DFA \( A \) is composite if its language can be decomposed into the intersection of the languages of smaller DFAs. More precisely, we say that \( A \) is \( k \)-factor composite if there exist \( k \) DFAs \( (A_i)_{1 \leq i \leq k} \) with less states than \( A \) such that \( L(A) = \bigcap_{i=1}^{k} L(A_i) \). We study the two following problems:

- **DFA DECOMP**
  - Given: DFA \( A \).
  - Question: Is \( A \) composite?

- **DFA BOUND-DECOMP**
  - Given: DFA \( A \) and integer \( k \in \mathbb{N} \).
  - Question: Is \( A \) \( k \)-factor composite?

The next example shows that decomposing DFAs can result in substantially smaller machines.

**Example.** Consider Figure 1. We simulate the interactions between a system and two clients by using finite words on the alphabet \( \{r_1, r_2, g_1, g_2, i\} \): At each time step, the system either receives a request from a client \((r_1, r_2)\), grants the open requests of a client \((g_1, g_2)\), or stays idle \((i)\). A basic property usually required is that every request is eventually granted. This specification is recognised by the DFA \( A \), which keeps track in its state of the current open requests, and only accepts if none is open when the input ends. Alternatively, this specification can be decomposed into the intersection of the languages defined by the DFAs \( A_1 \) and \( A_2 \): each one checks that the requests of the corresponding client are eventually granted. While in this precise example both ways of defining the specification are comparable, the latter scales dramatically better than the former when the number of clients increases: Suppose that there are now \( n \in \mathbb{N} \) clients. In order to check that all the requests are granted with a single DFA, we need \( 2^n \) states to keep track of all possible combinations of open requests, which is impractical when \( n \) gets too big. However, decomposing this specification into an intersection yields \( n \) DFAs of size two, one for each client. Note that, while in this specific example the decomposition is obvious, in general computing such a conjunctive form can be challenging: currently the best known algorithm needs exponential space.

![Figure 1 DFAs recognising specifications.](image-url)

The DFAs \( A_1 \) and \( A_2 \) check that every request of the first, resp. second, client is eventually granted. Given: DFA \( A \).

- Question: Is \( A \) composite?
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**Figure 2** Complexity of studied problems with containing classes, with our contribution in bold.

**DFAs in hardware.** Our considered problems are of great interest in hardware implementations of finite state machines [13] where realizing large DFAs poses a challenge [5]. In [2] the authors describe a state machine language for describing complex finite state hardware controllers, where the compiled state tables can automatically be input into a temporal logic model checker. If the control mechanism of the initial finite state machine can be split up into a conjunction of constraints, considering a decomposition instead could improve this workflow substantially. Decomposing a complex DFA $A$ can lead to a smaller representation of the DFA in total, as demonstrated in the previous example in Figure 1, and on top of that the individual smaller DFAs $A_i$ in the decomposition $L(A) = \cap_{i=1}^k L(A_i)$ can be placed independently on a circuit board, as they do not have to interact with each other and only need to read their common input from a global bus and signal acceptance as a flag to the bus. This allows for a great flexibility in circuit designs, as huge DFAs can be broken down into smaller blocks which fit into niches giving space for inflexible modules such as CPU cores.

**Reversible DFAs.** We focus our study on permutation DFAs, which are DFAs whose transition monoids are groups: each letter induces a one-to-one map from the state set into itself. These DFAs are also called reversible DFAs [8, 14]. Reversibility is stronger than determinism: this powerful property allows to deterministically navigate back and forth between the steps of a computation. This is particularly relevant in the study of the physics of computation, since irreversibility causes energy dissipation [10]. Remark that in the setting of DFAs, this power results in a loss of expressiveness: contrary to more powerful models (for instance Turing machines), reversible DFAs are less expressive than general DFAs.

**Related work.** The DFA DECOMP problem was first introduced in 2013 by Kupferman and Moscheiff [9]. They proved that it is decidable in EXPSPACE, but left open the exact complexity: the best known lower bound is hardness for NL. They gave more efficient algorithms for restricted domains: a PSPACE algorithm for permutation DFAs, and a PTIME algorithm for normal permutation DFAs, a class of DFAs that contains all commutative permutation DFAs. Recently, the DECOMP problem was proved to be decidable in LOGSPACE for DFAs with a singleton alphabet [7]. The trade-off between number and size of factors was studied in [12], where automata showing extreme behavior are presented, i.e., DFAs that can either be decomposed into a large number of small factors, or a small number of large factors.

**Contribution.** We expand the domain of instances over which the DECOMP problem is tractable. We focus on permutation DFAs, and we propose new techniques that improve the known complexities. All proofs omitted due to space restrictions can be found in the full version. Unless specified otherwise, the complexity of our algorithms do not depend on the size of the alphabet of the DFA. Our results, summarised by Figure 2, are presented as follows.
Section 3. We give an NP algorithm for permutation DFAs, and we show that the complexity is directly linked to the number of non-accepting states. This allows us to obtain a fixed-parameter tractable algorithm with respect to the number of non-accepting states (Theorem 1). Moreover, we prove that permutation DFAs with a prime number of states cannot be decomposed (Theorem 2).

Section 4. We consider commutative permutation DFAs, where the DECOMP problem was already known to be tractable, and we lower the complexity from PTIME to NL, and even LOGSPACE if the size of the alphabet is fixed (Theorem 9). While it is easy to decide whether a commutative permutation DFA is composite, we show that rich and complex behaviours still appear in this class: there exist families of composite DFAs that require polynomially many factors to get a decomposition. More precisely, we construct a family \((A_n^m)_{m,n\in\mathbb{N}}\) of composite DFAs such that \(A_n^m\) is a DFA of size \(n^m\) that is \((n-1)^{m-1}\)-factor composite but not \((n-1)^{m-1}-1\)-factor composite (Theorem 10). Note that, prior to this result, only families of composite DFAs with sublogarithmic width were known [7].

Section 5. Finally, we study the BOUND-DECOMP problem. High widths are undesirable for practical purposes: dealing with a huge number of small DFAs might end up being more complex than dealing with a single DFA of moderate size. The BOUND-DECOMP problem copes with this issue by limiting the number of factors allowed in the decompositions. We show that this flexibility comes at a cost: somewhat surprisingly, this problem is NP-complete for commutative permutation DFAs (Theorem 17), a setting where the DECOMP problem is easy. We also show that this problem is in PSPACE for the general setting (Theorem 16), and in LOGSPACE for unary DFAs i.e. with a singleton alphabet (Theorem 18).

2 Definitions

We denote by \(\mathbb{N}\) the set of non-negative integers \(\{0, 1, 2, \ldots\}\). For a word \(w = w_1w_2\ldots w_n\) with \(w_i \in \Sigma\) for \(1 \leq i \leq n\), we denote with \(w^R = w_n\ldots w_2w_1\) the reverse of \(w\). Moreover, for every \(\sigma \in \Sigma\), we denote by \(#_\sigma(w)\) the number of times the letter \(\sigma\) appears in \(w\). A natural number \(n > 1\) is called composite if it is the product of two smaller numbers, otherwise we say that \(n\) is prime. Two integers \(m, n \in \mathbb{N}\) are called co-prime if their greatest common divisor is 1. We will use the following well known results [6, 11]:

Bertrand’s Postulate. For all \(n > 3\) there is a prime number \(p\) satisfying \(n < p < 2n - 2\).

Bézout’s Identity. For every pair of integers \(m, n \in \mathbb{N}\), the set \(\{\lambda m - \mu n \mid \lambda, \mu \in \mathbb{N}\}\) contains exactly the multiples of the greatest common divisor of \(m\) and \(n\).

Deterministic finite automata. A deterministic finite automaton (DFA hereafter) is a 5-tuple \(A = (\Sigma, Q, q_I, \delta, F)\), where \(Q\) is a finite set of states, \(\Sigma\) is a finite non-empty alphabet, \(\delta : Q \times \Sigma \rightarrow Q\) is a transition function, \(q_I \in Q\) is the initial state, and \(F \subseteq Q\) is a set of accepting states. The states in \(Q \setminus F\) are called rejecting states. We extend \(\delta\) to words in the expected way, thus \(\delta : Q \times \Sigma^* \rightarrow Q\) is defined recursively by \(\delta(q, \varepsilon) = q\) and \(\delta(q, w_1w_2\ldots w_n) = \delta(\delta(q, w_1w_2\ldots w_{n-1}), w_n)\). The run of \(A\) on a word \(w = w_1\ldots w_n\) is the sequence of states \(s_0, s_1, \ldots, s_n\) such that \(s_0 = q_I\) and for each \(1 \leq i \leq n\) it holds that \(\delta(s_{i-1}, w_i) = s_i\). Note that \(s_n = \delta(q_I, w)\). The DFA \(A\) accepts \(w\) iff \(\delta(q_I, w) \in F\). Otherwise,
\( \mathcal{A} \) rejects \( w \). The set of words accepted by \( \mathcal{A} \) is denoted \( L(\mathcal{A}) \) and is called the \textit{language} of \( \mathcal{A} \). A language accepted by some DFA is called a \textit{regular language}.

We refer to the size of a DFA \( \mathcal{A} \), denoted \(|\mathcal{A}|\), as the number of states in \( \mathcal{A} \). A DFA \( \mathcal{A} \) is \textit{minimal} if every DFA \( \mathcal{B} \) such that \( L(\mathcal{B}) = L(\mathcal{A}) \) satisfies \(|\mathcal{B}| \geq |\mathcal{A}|\).

### Composite DFAs

We call a DFA \( \mathcal{A} \) \textit{composite} if there exists a family \( (\mathcal{B}_i)_{1 \leq i \leq k} \) of DFAs with \(|\mathcal{B}_i| < |\mathcal{A}|\) for all \( 1 \leq i \leq k \) such that \( L(\mathcal{A}) = \bigcap_{1 \leq i \leq k} L(\mathcal{B}_i) \) and call the family \( (\mathcal{B}_i)_{1 \leq i \leq k} \) a \textit{decomposition} of \( \mathcal{A} \). Note that, all \( \mathcal{B}_i \) in the decomposition satisfy \(|\mathcal{B}_i| < |\mathcal{A}|\) and \( L(\mathcal{A}) \subseteq L(\mathcal{B}_i) \). Such DFAs are called \textit{factors} of \( \mathcal{A} \), and \( (\mathcal{B}_i)_{1 \leq i \leq k} \) is also called a \( k \)-factor \textit{decomposition} of \( \mathcal{A} \). The width of \( \mathcal{A} \) is the smallest \( k \) for which there is a \( k \)-factor decomposition of \( \mathcal{A} \), and we say that \( \mathcal{A} \) is \textit{k-factor composite} if \( \text{width}(\mathcal{A}) \leq k \). We call a DFA \( \mathcal{A} \) \textit{prime} if it is not composite. We call a DFA \( \mathcal{A} \) \textit{trim} if all of its states are accessible from the initial state. As every non-\textit{trim} DFA \( \mathcal{A} \) is composite, we assume all given DFAs to be \textit{trim} in the following.

We call a DFA a \textit{permutation DFA} if for each letter \( \sigma \in \Sigma \), the function mapping each state \( q \) to the state \( \delta(q, \sigma) \) is a bijection. For permutation DFAs the transition monoid is a group. Further, we call a DFA \( \mathcal{A} = (\Sigma, Q, q_I, \delta, F) \) a \textit{commutative DFA} if \( \delta(q, uv) = \delta(q, vu) \) for every state \( q \) and every pair of words \( u, v \in \Sigma^* \). In the next sections we discuss the problem of being composite for the classes of permutation DFA, and commutative permutation DFAs.

### 3 Decompositions of Permutation DFAs

In this section, we study permutation DFAs. Our main contribution is an algorithm for the \textit{DECOMP} problem that is FPT with respect to the number of rejecting states:

\begin{itemize}
  \item \textbf{Theorem 1.} The \textit{DECOMP} problem for permutation DFAs is in NP. It is in FPT with parameter \( k \), being the number of rejecting states of DFA \( \mathcal{A} \), solvable in time \( O(2^k k^2 \cdot |\mathcal{A}|) \).
\end{itemize}

We prove Theorem 1 by introducing the notion of \textit{orbit-DFAs}: an orbit-DFA \( \mathcal{A}^U \) of a DFA \( \mathcal{A} \) is the DFA obtained by fixing a set of states \( U \) of \( \mathcal{A} \) as the initial state, and letting the transition function of \( \mathcal{A} \) act over it (thus the states of \( \mathcal{A}^U \) are subsets of the state space of \( \mathcal{A} \)). We prove three key results:

- A permutation DFA is composite if and only if it can be decomposed into its orbit-DFAs (Corollary 6);
- A permutation DFA \( \mathcal{A} \) can be decomposed into its orbit-DFAs if and only if for each of its rejecting states \( q \), there exists an orbit-DFA \( \mathcal{A}^U \) smaller than \( \mathcal{A} \) that \textit{covers} \( q \), that is, one of the states of \( \mathcal{A}^U \) contains \( q \) and no accepting states of \( \mathcal{A} \) (Lemma 7);
- Given a permutation DFA \( \mathcal{A} \) and a rejecting state \( q \), we can determine the existence of an orbit-DFA covering \( q \) in non-deterministic time \( O(|\mathcal{A}|^2) \), and in deterministic time \( O(2^k \cdot |\mathcal{A}|) \), where \( k \) is the number of rejecting states of \( \mathcal{A} \) (Lemma 8, (apx) Algorithm 1).

These results directly imply Theorem 1. We also apply them to show that the \textit{DECOMP} problem is trivial for permutation DFAs with a prime number of states.

\begin{itemize}
  \item \textbf{Theorem 2.} Let \( \mathcal{A} \) be a permutation DFA with at least one accepting state and one rejecting state. If the number of states of \( \mathcal{A} \) is prime, then \( \mathcal{A} \) is prime.
\end{itemize}

### 3.1 Proof of Theorem 1

Consider a DFA \( \mathcal{A} = (\Sigma, Q, q_I, \delta, F) \). We extend \( \delta \) to subsets \( U \subseteq Q \) in the expected way:

\[ \delta(U, w) = \{ q \in Q \mid q = \delta(p, w) \text{ for some } p \in U \} \] for every word \( w \in \Sigma^* \).
Figure 3 A DFA $A$ together with some of its orbit-DFAs. Accepting states are depicted in black, an orbit-DFA can be obtained by setting a subset containing a 1 as an initial state. For instance the orbit-DFAs $A^{(1,2,3)}$ and $A^{(1,5,6)}$ form a decomposition of $A$.

The orbit of $U$ is the collection $C_U = \{\delta(U,w) \subseteq Q \mid w \in \Sigma^*\}$ of subsets of $Q$ that can be reached from $U$ by the action of $\delta$. If the subset $U \subseteq Q$ contains the initial state $q_I$ of $A$, we define the orbit-DFA $A^U = (\Sigma, C_U, U, \delta, C')$, where the state space $C_U$ is the orbit of $U$, and the set $C'$ of accepting states is composed of the sets $U' \in C_U$ that contain at least one of the accepting states of $A$: $U' \cap F \neq \emptyset$. Note that $A^U$ can alternatively be defined as the standard subset construction starting with the set $U \subseteq Q$ as initial state. The definition of the accepting states guarantees that $L(A) \subseteq L(A^U)$.

**Proposition 3.** Every orbit-DFA $A^U$ of a DFA $A$ satisfies $L(A) \subseteq L(A^U)$.

**Example.** Let us detail the orbits of the DFA $A$ depicted in Figure 3. This DFA contains six states, and generates the following non-trivial orbits on its subsets of states:

- The 15 subsets of size 2 are split into two orbits: one of size 3, and one of size 12;
- The 20 subsets of size 3 are split into three orbits: two of size 4, and one of size 12;
- The 15 subsets of size 4 are split into two orbits, one of size 3, and one of size 12.

Figure 3 illustrates the four orbits smaller than $|A|$: they induce seven orbit-DFAs, obtained by setting as initial state one of the depicted subsets containing the initial state 1 of $A$.

In order to prove that a DFA is composite if and only if it can be decomposed into its orbit-DFAs, we prove that every factor $B$ of a permutation DFA $A$ can be turned into an orbit-DFA $A^U$ that is also a factor of $A$, and satisfies $L(A^U) \subseteq L(B)$. Our proof is based on a known result stating that factors can be turned into permutation DFAs:

**Lemma 4 ([9, Theorem 7.4]).** Let $A$ be a permutation DFA. For every factor $B$ of $A$, there exists a permutation DFA $C$ satisfying $|C| \leq |B|$ and $L(A) \subseteq L(C) \subseteq L(B)$.

We strengthen this result by showing how to transform factors into orbit-DFAs:

**Lemma 5.** Let $A$ be a permutation DFA. For every factor $B$ of $A$, there exists an orbit-DFA $A^U$ of $A$ satisfying $|A^U| \leq |B|$ and $L(A) \subseteq L(A^U) \subseteq L(B)$.

**Proof.** Let $A = (\Sigma, Q, q_I, \delta, F)$ be a permutation DFA, and let $B$ be a factor of $A$. By Lemma 4, there exists a permutation DFA $B' = (\Sigma, S, s_I, \eta, G)$ satisfying $|B'| \leq |B|$ and $L(A) \subseteq L(B') \subseteq L(B)$. We build, based on $B'$, an orbit-DFA $A^U$ of $A$ satisfying the statement.

We say that a state $q \in Q$ of $A$ is linked to a state $s \in S$ of $B'$, denoted $q \sim s$, if there exists a word $u \in \Sigma^*$ satisfying $\delta(q, u) = q$ and $\eta(s_I, u) = s$. Let $f : S \to 2^Q$ be the function mapping every state $s \in S$ to the set $f(s) \subseteq Q$ containing all the states $q \in Q$ that are linked to $s$ (i.e. satisfying $q \sim s$). We set $U = f(s_I)$. In particular, the initial state $q_I$ of $A$ is in $U$ since $\delta(q_I, \varepsilon) = q_I$ and $\eta(s_I, \varepsilon) = s_I$. We show that the orbit-DFA $A^U$ satisfies the desired conditions: $|A^U| \leq |B'|$ and $L(A) \subseteq L(A^U) \subseteq L(B')$. 
First, we show that $|A^U| \leq |B'|$ by proving that the function $f$ defined earlier maps $S$ surjectively into the orbit of $U$, which is the state space of $A^U$. Since both $A$ and $B'$ are permutation DFAs, we get that for all $q \in Q$, $s \in S$ and $a \in \Sigma$, then $q \sim s$ if and only if $\delta(q, a) \sim \eta(s, a)$ holds. Therefore, for every word $v \in \Sigma^*$, $f(\eta(s_1, v)) = \delta(f(s_1), v) = \delta(U, v)$. This shows that, as required, the image of the function $f$ is the orbit of $U$, and $f$ is surjective.

To conclude, we show that $L(A) \subseteq L(A^U) \subseteq L(B')$. Proposition 3 immediately implies that $L(A) \subseteq L(A^U)$. Therefore it is enough to show that $L(A^U) \subseteq L(B')$. Let $v \in L(A^U)$. By definition of an orbit-DFA, this means that the set $\delta(U, v)$ contains an accepting state $q_F$ of $A$. Since, as stated earlier, $\delta(q, v) = \delta(U, v)$, this implies (by definition of the function $f$) that the accepting state $q_F$ of $A$ is linked to $\eta(s_1, v)$, i.e., there exists a word $v' \in \Sigma^*$ such that $\delta(q, v') = q_F$ and $\eta(s_1, v') = \eta(s_1, v)$. Then $\delta(q_1, v') = q_F$ implies that $v'$ is in the language of $A$. Moreover, since $L(A) \subseteq L(B')$ by supposition, $v'$ is also accepted by $B'$, i.e., $\eta(s_1, v')$ is an accepting state of $B'$. Therefore, since $\eta(q_1, v') = \eta(q_1, v)$, the word $v'$ is also in the language of $B'$. This shows that $L(A^U) \subseteq L(B')$, which concludes the proof. ▶

As an immediate corollary, every decomposition of a permutation DFA can be transformed, factor after factor, into a decomposition into orbit-DFAs.

**Corollary 6.** A permutation DFA is composite if and only if it can be decomposed into its orbit-DFAs.

**Orbit cover.** Given a rejecting state $q \in Q \setminus F$ of $A$, we say that the orbit-DFA $A^U$ covers $q$ if $|A^U| < |A|$, and $A^U$ contains a rejecting state $U' \subseteq Q$ that contains $q$. Remember that, by definition, this means that $U'$ contains no accepting state of $A$, i.e., $U' \cap F = \emptyset$. We show that permutation DFAs that can be decomposed into their orbit-DFAs are characterized by the existence of orbit-DFAs covering each of their rejecting states.

**Lemma 7.** A permutation DFA $A$ is decomposable into its orbit-DFAs if and only if every rejecting state of $A$ is covered by an orbit-DFA $A'$ of $A$ satisfying $|A'| < |A|$.  

**Proof.** Let $A = (\Sigma, Q, q_I, \delta, F)$ be a permutation DFA. We prove both implications.

Suppose that $A$ can be decomposed into its orbit-DFAs $(A^{U_i})_{1 \leq i \leq k}$, and let $q \in Q \setminus F$ be a rejecting state of $A$. We show that $q$ is covered by every orbit-DFA $A^{U_i}$ that rejects a word $w \in \Sigma^*$ satisfying $\delta(q_1, w) = q$. Formally, let $w \in \Sigma^*$ be a word satisfying $\delta(q_1, w) = q$. Then $w \notin L(A) = \bigcap_{i=1}^k L(A^{U_i})$, hence there exists $1 \leq i \leq n$ such that $w \notin L(A^{U_i})$. Let $U' \subseteq Q$ be the state visited by $A^{U_i}$ after reading $w$. Then, by applying the definition of an orbit-DFA, we get that $q \in U'$ since $\delta(q_1, w) = q$, and $U' \cap F = \emptyset$ since $U'$ is a rejecting state of $A^{U_i}$ (as $w \notin L(A^{U_i})$). Therefore, $A^{U_i}$ covers $q$. Moreover, $|A^{U_i}| < |A|$ since $A^{U_i}$ is a factor of $A$.

Conversely, let us fix an enumeration $q_1, q_2, \ldots, q_m$ of the rejecting states of $A$, and suppose that for all $1 \leq i \leq m$ there is an orbit-DFA $A^{U_i}$ of $A$ that covers $q_i$ and satisfies $|A^{U_i}| < |A|$. Let $(U_{i,j})_{1 \leq j \leq n_i}$ be an enumeration of the subsets in the orbit of $U_i$ that contain the initial state $q_i$ of $A$. We conclude the proof by showing that $S = \{A^{U_{i,j}} \mid 1 \leq i \leq m, 1 \leq j \leq n_i\}$ is a decomposition of $A$. Note that we immediately get $|A^{U_{i,j}}| = |A^{U_i}| < |A|$ for all $1 \leq i \leq m$ and $1 \leq j \leq n_i$. Moreover, Proposition 3 implies $L(A) \subseteq \bigcap_{A' \in S} L(A')$. To complete the proof, we show that $\bigcap_{A' \in S} L(A') \subseteq L(A)$. Let $w \in \Sigma^*$ be a word rejected by $A$. To prove the desired inclusion, we show that there is a DFA $A' \in S$ that rejects $w$. Since $w \notin L(A)$, the

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1 Remark that for general DFAs we only get that $q \sim s$ implies $\delta(q, a) \sim \eta(s, a)$ from the determinism. It is the backward determinism of the permutation DFAs $A$ and $B'$ that gives us the reverse implication.
run of $\mathcal{A}$ on $w$ starting from the initial state ends in a rejecting state $q_i$, for some $1 \leq i \leq m$. By supposition the orbit-DFA $\mathcal{A}^{U_i}$ covers $q_i$, hence the orbit of $U_i$ contains a set $U' \subseteq Q$ that contains $q_i$ and no accepting state. Note that there is no guarantee that $\mathcal{A}^{U_i}$ rejects $w$: while the set $\delta(U_i, w)$ contains $q_i$, it is not necessarily equal to $U'$, and might contain accepting states. Hence, as $\mathcal{A}$ is a permutation DFA, we can reverse all of the transitions of $\mathcal{A}$ to get a path labeled by the reverse of $w$ that starts from $U'$ (that contains $q_i$), and ends in one of the sets $U_{i,j}$ (that contains $q_j$). Therefore, by reversing this path back to normal, we get that $\delta(U_{i,j}, w) = U''$, hence the orbit-DFA $\mathcal{A}^{U_{i,j}} \in S$ rejects $w$. Therefore, every word rejected by $\mathcal{A}$ is rejected by an orbit-DFA $\mathcal{A}' \in S$, which shows that $\bigcap_{\mathcal{A}' \in S} L(\mathcal{A}') \subseteq L(\mathcal{A})$. ▶

This powerful lemma allows us to easily determine whether a permutation DFA is composite if we know its orbits. For instance, the DFA $\mathcal{A}$ depicted in Figure 3 is composite since the orbit-DFA $\mathcal{A}^{(1,2,3)}$ covers its five rejecting states. Following the proof of Lemma 7, we get that $(\mathcal{A}^{(1,2,3)}, \mathcal{A}^{(1,5,6)})$ is a decomposition of $\mathcal{A}$, and so is $(\mathcal{A}^{(1,2,6)}, \mathcal{A}^{(1,3,5)})$.

To conclude, we give an algorithm checking if a rejecting state is covered by an orbit-DFA.

Lemma 8. Given a permutation DFA $\mathcal{A}$ and a rejecting state $q$, we can determine the existence of an orbit-DFA that covers $q$ in nondeterministic time $O(k \cdot |\mathcal{A}|^2)$, and in deterministic time $O(2^k k \cdot |\mathcal{A}|^2)$, where $k$ is the number of rejecting states of $\mathcal{A}$.

Proof. We can decide in $\text{NP}$ whether there exists an orbit-DFA $\mathcal{A}^{U}$ of $\mathcal{A}$ that covers $p$: we non-deterministically guess among the set of rejecting states of $\mathcal{A}$ a subset $U'$ containing $p$. Then, we check in polynomial time that the orbit of $U'$ is smaller than $|\mathcal{A}|$. This property can be checked in time $O(|\mathcal{A}|^2)$. Since $\mathcal{A}$ is trim, in the orbit of $U'$ there is a set $U$ containing the initial state of $\mathcal{A}$. Moreover, since $\mathcal{A}$ is a permutation DFA, $U$ and $U'$ induce the same orbit. Hence, $p$ is covered by the orbit-DFA $\mathcal{A}^{U}$. Finally, we can make this algorithm deterministic by searching through the $2^k$ possible subsets $U'$ of the set of rejecting states of $\mathcal{A}$. ▶

3.2 Proof of Theorem 2

Thanks to the notion of orbit DFAs we are able to prove that a permutation DFA which has a prime number of states with at least one accepting and one rejecting, is prime.

Proof. Let $\mathcal{A} = (\Sigma, Q, q_I, \delta, F)$ be a trim permutation DFA with a state space $Q$ of prime size that contains at least one accepting state and one rejecting state. We show that the only orbit of $\mathcal{A}$ smaller than $|Q|$ is the trivial orbit $\{Q\}$. This implies that $\mathcal{A}$ cannot be decomposed into its orbit-DFAs, which proves that $\mathcal{A}$ is prime by Lemma 5.

Let us consider a strict subset $U_1 \neq \emptyset$ of the state space $Q$, together with its orbit $\mathcal{C}_{U_1} = \{U_1, U_2, \ldots, U_m\}$. We prove that $m \geq |Q|$. First, we show that all the $U_i$ have the same size: since $U_1$ is an element of the orbit of $U_1$, there exists a word $u_i \in \Sigma^*$ satisfying $\delta(U_i, u_i) = U_1$, and, as every word in $\Sigma^*$ induces via $\delta$ a permutation on the state space, $|U_i| = |\delta(U_i, u_i)| = |U_1|$. Second, for every $q \in Q$, we define the multiplicity of $q$ in $\mathcal{C}_{U_1}$ as the number $\lambda(q) \in \mathbb{N}$ of distinct elements of $\mathcal{C}_{U_1}$ containing the state $q$. We show that all the states $q$ have the same multiplicity: since $\mathcal{A}$ is trim, there exists a word $u_q \in \Sigma^*$ satisfying $\delta(q_1, u_q) = q$, hence $u_q$ induces via $\delta$ a bijection between the elements of $\mathcal{C}_{U_1}$ containing $q_1$ and those containing $q$, and $\lambda(q) = \lambda(\delta(q_1, u_q)) = \lambda(q_1)$. By combining these results, we obtain $m = |U_1| = \sum_{q \in Q} |U_i| = \sum_{q \in Q} \lambda(q) = \lambda(q_1) \cdot |Q|$. Therefore, as $|Q|$ is prime by supposition, either $m$ or $|U_1|$ is divisible by $|Q|$. However, $|U_1| \leq Q$, hence $|U_1| < |Q|$, which shows that $m$ is divisible by $|Q|$. In particular, we get $m \geq |Q|$, which concludes the proof. ▶

2. Remark that, if $\mathcal{A}$ is not a permutation DFA, then some states might not have incoming transitions for every letter. Thus, the reversal of $w$ might not be defined.
4 Decompositions of Commutative Permutation DFAs

We now study commutative permutation DFAs: a DFA $A = \langle \Sigma, Q, q_1, \delta, F \rangle$ is commutative if $\delta(q, uv) = \delta(q, vu)$ for every state $q$ and every pair of words $u, v \in \Sigma^*$. Our main contribution is an NL algorithm for the DECOMP problem for commutative permutation DFAs. Moreover, we show that the complexity goes down to LOGSPACE for alphabets of fixed size.

Theorem 9. The DECOMP problem for commutative permutation DFAs is in NL, and in LOGSPACE when the size of the alphabet is fixed.

The proof of Theorem 9 is based on the notion of covering word: a word $w \in \Sigma^*$ covers a rejecting state $q$ of a DFA $A = \langle \Sigma, Q, q_1, \delta, F \rangle$ if $\delta(q, w) \neq q$, and for every $\lambda \in \mathbb{N}$, the state $\delta(q, w^\lambda)$ is rejecting. We prove two related key results:

\begin{itemize}
\item A commutative permutation DFA is composite if and only if each of its rejecting states is covered by a word (Lemma 12).
\item We can decide in NL (LOGSPACE when the size of the alphabet is fixed) if a given rejecting state of a DFA is covered by a word (Lemma 13, and Algorithm 2 in appendix).
\end{itemize}

These results immediately imply Theorem 9. We conclude this section by showing an upper bound on the width and constructing a family of DFAs of polynomial width.

Theorem 10. The width of every composite permutation DFA is smaller than its size. Moreover, for all $m, n \in \mathbb{N}$ such that $n$ is prime, there exists a commutative permutation DFA of size $n^m$ and width $(n - 1)^{m-1}$. We show that the width of a commutative permutation DFA is bounded by its number of rejecting states (Lemma 12). Then, for each $m, n \in \mathbb{N}$ with $n$ prime, we define a DFA $A_n^m$ of size $n^m$ that can be decomposed into $(n - 1)^{m-1}$ factors (Proposition 14), but not into $(n - 1)^{m-1} - 1$ (Proposition 15).

4.1 Proof of Theorem 9

The proof is based on the following key property of commutative permutation DFAs: In a permutation DFA $A$, every input word acts as a permutation on the set of states, generating disjoint cycles, and if $A$ is commutative these cycles form an orbit.

Proposition 11. Let $A = \langle \Sigma, Q, q_1, \delta, F \rangle$ be a commutative permutation DFA. For all $u \in \Sigma^*$, the sets $\{ \delta(q, u^\lambda) \mid \lambda \in \mathbb{N} \}$ partition $Q$ and form an orbit of $A$.

Proof. Let $A = \langle \Sigma, Q, q_1, \delta, F \rangle$ be a commutative permutation DFA. Given $u \in \Sigma^*$ and $q \in Q$, the sequence of states $\delta(q, u), \delta(q, u^2), \ldots, \delta(q, u^p)$ visited by applying $\delta$ on iterations of $u$ eventually repeats i.e. $\delta(q, u^x) = \delta(q, u^y) = p$ for some $x, y \in \mathbb{N}$ and $p \in Q$. Since $A$ is a permutation DFA, it is both forward and backward deterministic, thus the set of visited states $\{ \delta(q, u^\lambda) \mid \lambda \in \mathbb{N} \}$ is a cycle that contain both $p$ and $q$. The collection $\{ \delta(q, u^\lambda) \mid \lambda \in \mathbb{N} \}$ forms an orbit of $A$ by commutativity. Formally, for all $u, v \in \Sigma^*$ and every $q \in Q$, we have: $\delta(\delta(q, u^\lambda) \mid \lambda \in \mathbb{N}), v) = \{ \delta(q, u^\lambda) \mid \lambda \in \mathbb{N} \} = \{ \delta(q, u^\lambda) \mid \lambda \in \mathbb{N} \}$. ▶

We proved with Corollary 6 and Lemma 7 that a permutation DFA is composite if and only if each of its rejecting states is covered by an orbit-DFA. We now reinforce this result for commutative permutation DFAs. As stated before, we say that a word $u \in \Sigma^*$ covers a rejecting state $q$ of a DFA $A = \langle \Sigma, Q, q_1, \delta, F \rangle$ if $u$ induces from $q$ a non-trivial cycle composed of rejecting states: $\delta(q, u) \neq q$, and $\delta(q, u^\lambda)$ is rejecting for all $\lambda \in \mathbb{N}$. Note that the collection $\{ \delta(q, u^\lambda) \mid \lambda \in \mathbb{N} \}$ forms an orbit of $A$ by Proposition 11. We show that we can determine if $A$ is composite by looking for words covering its rejecting states.
Lemma 12. For every \( k \in \mathbb{N} \), a commutative permutation DFA \( A \) is \( k \)-factor composite if and only if there exist \( k \) words that, together, cover all the rejecting states of \( A \).

Proof. Let \( A = (\Sigma, Q, q_0, \delta, F) \) be a commutative permutation DFA and \( k \in \mathbb{N} \). We start by constructing \( k \) factors based on \( k \) covering words. Suppose that there exist \( k \) words \( u_1, u_2, \ldots, u_k \) such that every rejecting state \( q \in Q \setminus F \) is covered by one of the \( u_i \). Note that all the \( u_i \) covering at least one state \( q \) do not act as the identity on \( Q \) (since \( \delta(q, u_i) \neq q \)), therefore we suppose, without loss of generality, that none of the \( u_i \) acts as the identity on \( Q \). For every \( 1 \leq i \leq k \), let \( U_i = \{\delta(q, u_i^\lambda) | \lambda \in \mathbb{N}\} \). We show that \( (A_U^i)_{1 \leq i \leq k} \) is a decomposition of \( A \). As none of the \( u_i \) acts as the identity on \( Q \), Proposition 11 implies that every \( A_U^i \) is smaller than \( A \). Moreover, Proposition 3 implies that \( L(A) \subseteq \bigcap_{i=1}^k L(A_U^i) \). To conclude, we show that \( \bigcap_{i=1}^k L(A_U^i) \subseteq L(A) \). Let \( u \notin L(A) \). By supposition, there exists \( 1 \leq i \leq k \) such that \( u_i \) covers \( \delta(q, u) \). As a consequence, the set

\[
\delta(U_i, u) = \delta(\{\delta(q, u_i^\lambda) | \lambda \in \mathbb{N}\}, u) = \{\delta(q, u_i^\lambda) | \lambda \in \mathbb{N}\} = \{\delta(q, u u_i^\lambda) | \lambda \in \mathbb{N}\}
\]

contains no accepting state of \( A \), hence it is a rejecting state of \( A_U^i \). As a consequence, we get \( u \notin L(A_U^i) \subseteq \bigcap_{i=1}^k L(A_U^i) \), which proves that \( \bigcap_{i=1}^k L(A_U^i) \subseteq L(A) \).

We now construct \( k \) factors based on \( k \) covering words. Suppose that \( A \) has a \( k \)-factor decomposition \((B_i)_{1 \leq i \leq k}\). Lemma 4 directly implies that this decomposition can be transformed into a decomposition \((C_i)_{1 \leq i \leq k}\) of \( A \), where \( C_i = (\Sigma, S_i, s_i, \eta_i, G_i) \) are permutation DFAs. For every \( 1 \leq i \leq k \), we build a word \( u_i \) based on \( C_i \), we prove that every rejecting state of \( A \) is covered by one of these \( u_i \). Consider \( 1 \leq i \leq k \). Since \( C_i \) is a factor of \( A \), in particular \( |C_i| < |A| \), hence there exist two input words \( v_i, w_i \in \Sigma^* \) such that \( A \) reaches different states on \( v_i \) and \( w_i \), but \( C_i \) reaches the same state: \( \delta(q_i, v_i) \neq \delta(q_i, w_i) \) but \( \eta_i(s_i, v_i) = \eta_i(s_i, w_i) \). Note that both \( A \) and \( C_i \) are permutation DFAs, hence there exists a power \( u_i^{\eta_i} \) of \( v_i \) that induces the identity function on both state spaces \( Q \) and \( S_i \). We set \( u_i = v_i^{\eta_i} \), which guarantees that:

\[
\delta(q_i, u_i) = \delta(q_i, v_i^{\eta_i}) \neq \delta(q_i, v_i) = \eta_i(s_i, v_i) = \eta_i(s_i, u_i^{\eta_i}) = \eta_i(s_i, v_i^{\eta_i}) = q_i;
\]

in other words, \( u_i \) moves the initial state \( q_i \) of \( A \), but fixes the initial state \( s_i \) of \( C_i \).

We now prove that each rejecting state of \( A \) is covered by one of the \( u_i \). Let \( q \in Q \setminus F \) be a rejecting state of \( A \). Since \( A \) is trim, there exists a word \( u_q \in \Sigma^* \) such that \( \delta(q, u_q) = q \). Then, as \( u_q \notin L(A) \) and \( (C_i)_{1 \leq i \leq k} \) is a decomposition of \( A \), there exists \( 1 \leq i \leq k \) such that \( u_q \notin L(C_i) \). We show that the word \( u_i \) covers the rejecting state \( q \): we prove that \( \delta(q, u_i) \neq q \), and that \( \delta(q, u_i) \) is rejecting for every \( \lambda \in \mathbb{N} \). First, since \( A \) is a commutative permutation DFA and \( u_i \) moves \( q_i \), we get that \( \delta(q, u_i) = \delta(q_i, u_i u_i) = \delta(q_i, u_i u_i) = q_i = q \). Moreover, for all \( \lambda \in \mathbb{N} \), since \( u_q \notin L(C_i) \) by supposition and \( u_i \) fixes \( s_i \), the DFA \( C_i \) also rejects the word \( u_i^{\eta_i} u_q \). Therefore, as \( L(A) \subseteq L(C_i) \), we finally get that \( \delta(q, u_i) = \delta(q_i, u_i u_i) = \delta(q_i, u_i u_i) = q_i = q \).}

By Lemma 12, to conclude the proof of Theorem 9 we show that we can decide in \( \text{NL} \) (and in \( \text{LOGSPACE} \) when the size of the alphabet is fixed) whether a given rejecting state of a DFA is covered by a word (since in the \text{DECOMP} problem we can afford to pick a covering word for each state). As we consider commutative permutation DFAs, we can represent a covering word by the number of occurrences of each letter, which are all bounded by \(|Q|\).

Lemma 13. Let \( A \) be a commutative permutation DFA and \( p \) a rejecting state.

1. We can determine the existence of a word covering \( p \) in space \( \mathcal{O}(|\Sigma| \cdot \log |Q|) \);
2. We can determine the existence of a word covering \( p \) in \( \text{NL} \)
We finally study the guessing setting, we show that the problem is in non-trivial orbit-DFAs. Final states are depicted in black.

4.2 Proof of Theorem 10

As a direct consequence of Lemma 12, the width of every commutative permutation DFA $A$ is bounded by the number of rejecting states of $A$, hence, it is smaller than $|A|$. To conclude the proof of Theorem 10, for all $m, n \in \mathbb{N}$ with $n$ prime, we define a DFA $A_n^m$ of size $n^m$ and width $(n - 1)^{m-1}$ on the alphabet $\Sigma = \{a_1, a_2, \ldots, a_m\}$. For all $\ell \in \mathbb{N}$, let $[\ell]$ denote the equivalence class of $\ell$ modulo $n$. Let $L_n^m \subseteq \Sigma^*$ be the language composed of the words $w$ such that for at least one letter $a_i \in \Sigma$ the number $\#a_i(w)$ of $a_i$ in $w$ is a multiple of $n$, and for at least one other letter $a_j \in \Sigma$, the number $\#a_j(w)$ of $a_j$ in $w$ is not a multiple of $n$:

$$L_n^m = \{ w \in \Sigma^* \mid \#a_i(w) = [0] \text{ and } \#a_j(w) \neq [0] \text{ for some } 1 \leq i, j \leq m \}.$$ 

The language $L_n^m$ is recognised by a DFA $A_n^m$ of size $n^m$ that keeps track of the value modulo $n$ of the number of each $a_i$ already processed. The state space of $A_n^m$ is the direct product $(\mathbb{Z}/n\mathbb{Z})^m$ of $m$ copies of the cyclic group $\mathbb{Z}/n\mathbb{Z} = \{0, [1], \ldots, [n-1]\}$; the initial state is $([0], [0], \ldots, [0])$; the final states are the ones containing at least one component equal to $[0]$ and one component distinct from $[0]$; and the transition function increments the $i$th component when an $a_i$ is read: $\delta(([j_1], [j_2], \ldots, [j_m]), a_i) = ([j_1], [j_2], \ldots, [j_{i-1}], [j_i+1], [j_{i+1}], \ldots, [j_m])$.

Figure 4 illustrates the particular case $n = 5$ and $m = 2$.

To prove that the width of $A_n^m$ is $(n - 1)^{m-1}$, we first show that the $(n - 1)^{m-1}$ words $\{a_1^{\lambda_1}a_2^{\lambda_2}\ldots a_m^{\lambda_m} \mid 1 \leq \lambda_i \leq n - 1 \}$ cover all the rejecting states, thus by Lemma 12:

**Proposition 14.** The DFA $A_n^m$ is $(n - 1)^{m-1}$-factor composite.

Then, we prove that there exist no word that covers two states among the $(n - 1)^{m-1}$ rejecting states $\{([1],[k_2],[k_3],\ldots,[k_m]) \mid 1 \leq k_i \leq m - 1 \}$. Therefore, we need at least $(n - 1)^{m-1}$ words to cover all of the states, thus by Lemma 12:

**Proposition 15.** The DFA $A_n^m$ is not $((n - 1)^{m-1} - 1)$-factor composite.

5 Bounded Decomposition

We finally study the **BOUND-DECOMP** problem: Given a DFA $A$ and an integer $k \in \mathbb{N}$ encoded in unary, can we determine whether $A$ is decomposable into $k$ factors? For the general setting, we show that the problem is in PSPACE: it can be solved by non-deterministically guessing $k$ factors, and checking that they form a decomposition.
Theorem 16. The Bound-Decomp problem is in PSPACE.

For commutative permutation DFAs, we obtain a better algorithm through the use of the results obtained in the previous sections, and we show a matching hardness result.

Theorem 17. The Bound-Decomp problem for commutative permutation DFAs is NP-complete.

Both parts of the proof of Theorem 17 are based on Lemma 12: a commutative permutation DFA is k-factor composite if and only if there exist k words covering all of its rejecting states. We prove the two following results:

- Bounded compositionality is decidable in NP, as it is sufficient to non-deterministically guess a set of k words, and check whether they cover all rejecting states (Lemma 19);
- The NP-hardness is obtained by reducing the Hitting Set problem, a well known NP-complete decision problem. We show that searching for k words that cover the rejecting states of a DFA is as complicated as searching for a hitting set of size k (Lemma 20).

We finally give a LOGSPACE algorithm based on known results for DFAs on unary alphabets [7].

Theorem 18. The Bound-Decomp problem for unary DFAs is in LOGSPACE.

5.1 Proof of Theorem 17

By Lemma 12, a commutative permutation DFA A is k-factor composite if and only if its rejecting states can be covered by k words. As we can suppose that covering words have size linear in |A| (see proof of Lemma 13), the Bound-Decomp problem is decidable in NP: we guess a set of k covering words and check in polynomial time if they cover all rejecting states.

Lemma 19. The Bound-Decomp problem for commutative permutation DFAs is in NP.

We show that the problem is NP-hard by a reduction from the Hitting Set problem.

Lemma 20. The Bound-Decomp problem is NP-hard for commutative permutation DFAs.

Proof. The proof goes by a reduction from the Hitting Set problem (HIT for short), known to be NP-complete [4]. The HIT problem asks, given a finite set \( S = \{1, 2, \ldots, n\} \subseteq \mathbb{N} \), a finite collection of subsets \( F = \{C_1, C_2, \ldots, C_m\} \subseteq 2^S \), and an integer \( k \in \mathbb{N} \), whether there
and for which all states are accepting i.e. rejecting states with the same emphasized sector. Since in Figure 5, each instance of HIT with \( S = \{1, 2\} \) and \( \mathcal{F} = \{\{1\}, \{1, 2\}, \{2\}\} \) using \( \mu = 3 \) and \( \tau = 5 \). Accepting states are filled black while rejecting states are sectored.

is a subset \( X \subseteq S \) with \( |X| \leq k \) and \( X \cap C_i \neq \emptyset \) for all \( 1 \leq i \leq m \). We describe how to construct a DFA \( \mathcal{A} = (\Sigma, Q, \delta, F) \) that is \((k + 1)\)-factor composite if and only if the HIT instance \((S, \mathcal{F}, k)\) has a solution.

**Automaton construction.** To be constructed, the automaton \( \mathcal{A} \) requires \( \mu, \tau \) defined as the smallest prime numbers that fulfill \( n < \mu \) and \( m < \tau \) and \( 2 < \mu < \tau \). By Bertrand’s postulate [11], \( \mu \) and \( \tau \) have the same value polynomial in \( m + n \). The state space of \( \mathcal{A} \) is defined as \( Q = \{0, 1, \ldots, \mu - 1\} \times \{0, 1, \ldots, \tau - 1\} \times \{0, 1\} \) with \( q_1 = (0, 0, 0, 0) \) as initial state. Let us define the subset of states \( Q_\perp = \{q_1, q_2, q_3, q_4\} \subseteq Q \) to encode instances of HIT and the subset \( Q_\top = \{q_1, q_2, q_3, q_4\} \subseteq Q \) which is a copy of \( Q_\perp \) with minor changes. The example in Figure 5 gives some intuition on the construction of \( \mathcal{A} \).

The DFA \( \mathcal{A} \) is defined over the alphabet \( \Sigma = \{a, b, c, d\} \) with the transition function defined for each state \( q = (q_1, q_2, q_3, q_4) \) by \( \delta(q, a) = (q_1 + 1 \mod \mu, q_2, q_3, q_4) \), \( \delta(q, b) = (q_1, q_2 + 1 \mod \mu, q_3, q_4) \), \( \delta(q, c) = (q_1, q_2, q_3 + 1 \mod \tau, q_4) \) and \( \delta(q, d) = (q_1, q_2, q_3, q_4 + 1 \mod 2) \). Note that, \( \mathcal{A} \) can be seen as a product of four prime finite fields. In particular, for every \( q_3 \in \{0, \ldots, \tau - 1\} \) the subset of states \( \{(x, y, q_3, 0) \in Q_\perp \mid 0 \leq x, y \leq \mu - 1\} \) can be seen as the direct product of two copies of the field of order \( \mu \) (a.k.a. \( \mathbb{F}_\mu \)), thus inheriting the structure of a \( \mathbb{F}_\mu \)-vector space of origin \( (0, 0, q_3, 0) \). We use these \( \tau \)-disjoint vector spaces to represent the collections of \( \mathcal{F} \) thanks to the acceptance of states. More precisely, each collection \( C_i \in \mathcal{F} \) is encoded through the vector space \( \{(x, y, i, 0) \in Q_\perp \mid 1 \leq i \leq m\} \) and each \( v \in C_i \) is encoded by the non-acceptance of all states belonging to the line \( \{(x, y, i, 0) \in Q_\perp \mid y = vx \mod \mu\} \).

In Figure 5, each \( C_i \) is presented by an instance of \( \mathbb{F}_3 \times \mathbb{F}_3 \) and each \( v \in C_i \) is depicted by rejecting states with the same emphasized sector. Since \( \tau > m \), there are extra vector spaces for which all states are acceptable i.e. \( \{(q_1, q_2, q_3, 0) \in Q_\perp \mid q_3 \notin \{1, 2, \ldots, m\}\} \subseteq \mathcal{F} \). The acceptance of states of \( Q_\top \) is defined similarly as for \( Q_\perp \) except that the origins of vector spaces are accepting in \( Q_\top \) (see Figure 5). Formally, the rejecting states of \( \mathcal{A} \) is defined by \( \mathcal{F} = R_\perp \cup R_\top \) where \( R_\perp = \{(q_1, q_2, q_3, 0) \in Q_\perp \mid q_2 = vq_1 \mod \mu, 1 \leq q_3 \leq m, v \in C_{q_3}\} \) and \( R_\top = \{(q_1, q_2, q_3, 1) \in Q_\top \mid (q_1, q_2, q_3, 0) \in R_\perp, q_1 \neq 0, q_2 \neq 0\} \). All other states are accepting, i.e., we set \( F = Q \setminus \mathcal{F} \). So, the acceptance of the subsets of states \( Q_\perp \) and \( Q_\top \) only differ by \( O \cap Q_\perp \subseteq \mathcal{F} \) and \( O \cap Q_\top \subseteq \mathcal{F} \) where \( O = \{(0, 0, q_3, 0) \in Q \mid q_3 \in \{1, \ldots, m\}\} \).

The cornerstone which holds the connection between the two problems is the way the rejecting states of \( O \) can be covered. In fact, since \( Q_\top \) mimics \( Q_\perp \) for states in \( Q \setminus O \), all rejecting states of \( Q \setminus O \) can be covered by the single word \( d \in \Sigma \). In addition, most words do
not cover any rejecting states of $A$, as stated by the following claim. Hereafter, we say that a word $w \in \Sigma^*$ is concise when it satisfies $\#_\sigma(w) < h_\sigma$ for all $\sigma \in \Sigma$, where $h_\sigma \in \{2, \mu, \tau\}$ is the size of the cycle induced by $\sigma$.

\[\triangleleft\] Claim 21. Let $u \in \Sigma^*$ be a concise word that covers some rejecting state of $A$:
1. $u$ must belong either in $\{d\}^*$ or in $\{a, b\}^* \setminus (\{a\}^* \cup \{b\}^*)$.
2. $u$ covers some rejecting state of $Q_\tau$ iff $u$ covers all rejecting states of $Q_\tau$ if $u = d$.
3. $u$ covers $(0, 0, i, 0) \in \Omega$ iff $u \in \{a, b\}^*$ and $\#_a(u) \equiv v \cdot \#_a(u) \mod \mu$ for some $v \in C_i$.

Proof of item 1. The statement is a direct consequence of the following:

i. Every concise word $u$ satisfying $\#_c(u) > 0$ covers no rejecting state of $A$;
ii. Every concise word $u \in \{a\}^* \cup \{b\}^*$ covers no rejecting state of $A$;
iii. Every concise word $u$ satisfying $\#_a(u) > 0$ and $\#_d(u) > 0$ covers no rejecting state of $A$;
iv. Every concise word $u$ satisfying $\#_a(u) > 0$ and $\#_d(u) > 0$ covers no rejecting state of $A$.

In order to prove these four properties, we now fix a state $q = (q_1, q_2, q_3, q_4) \in \Omega$, and we show that, in each case, iterating a word of the corresponding form starting from $q$ will eventually lead to an accepting state:

(i.) Let $u$ be a concise word satisfying $\#_c(u) > 0$. Since $u$ is concise we have $\#_a(u) < \tau$. Hence, as $\tau$ is prime, there exists $\lambda \in \mathbb{N}$ such that $\lambda \cdot \#_c(u) \equiv -q_1 \mod \tau$. Therefore the third component of $\delta(q, u^\lambda)$ is $0$, thus it is an accepting state of $A$.

(ii.) Let $u \in \{a\}^*$ be a concise word (if $u \in \{b\}^*$ instead, the same proof works by swapping the roles of $q_1$ and $q_2$). Since $u$ is concise we have $0 < \#_a(u) < \mu$. Hence, as $\mu$ is prime there exists $\lambda_1, \lambda_2 \in \mathbb{N}$ satisfying $\lambda_1 \cdot \#_a(u) \equiv -q_1 \mod \mu$ and $\lambda_2 \cdot \#_a(u) \equiv -q_1 + 1 \mod \mu$. Therefore, if $q_2 \neq 0$, we get that $\delta(q, u^{\lambda_1}) = (0, q_2, q_3, q_4)$ is an accepting state of $A$, and if $q_2 = 0$, we get that $\delta(q, u^{\lambda_2}) = (1, 0, q_3, q_4)$ is an accepting state of $A$.

(iii.) Let $u$ be a concise word satisfying $\#_a(u) > 0$ and $\#_d(u) > 0$. Since $\mu$ is a prime number greater than 2, there exist $\alpha \in \mathbb{N}$ such that $\mu - 2\alpha = 1$, thus $2\alpha \equiv -1 \mod \mu$. Moreover, since $u$ is concise we have $\#_d(u) = 1$ and $\#_a(u) < \mu$. Hence there exists $\beta \in \mathbb{N}$ such that $\beta \cdot \#_a(u) \equiv 1 \mod \mu$. Therefore, if we let $\lambda = 2\alpha \beta q_1 + \mu(1 - p_4)$, we get

\[
\#_a(u^\lambda) = 2\alpha \cdot \beta \cdot \#_a(u) \cdot q_1 + \mu(1 - p_4) \cdot \#_a(u) \equiv -q_1 \mod \mu;
\]

\[
\#_d(u^\lambda) = 2\alpha \beta q_1 + \mu \cdot (1 - p_4) \equiv 1 - p_4 \mod 2;
\]

As a consequence, the first component of $\delta(q, u^\lambda)$ is $0$ and its fourth component is $1$, hence it is an accepting state of $A$.

(iv.) Let $u$ be a concise word satisfying $\#_a(u) > 0$ and $\#_d(u) > 0$. Then we can prove that $u$ does not cover $q$ as in point (3), by swapping the roles of $q_1$ and $q_2$. \(\triangleleft\)

Proof of item 2. First, remark that $d$ is the only concise word of $\{d\}^*$. By construction of $A$, we have $(q_1, q_2, q_3, 0) \in F$ if and only if $(q_1, q_2, q_3, 1) \in F$ holds for all $(q_1, q_2, q_3, q_4) \in \Omega \setminus O$. Thus, for all $(q_1, q_2, q_3, q_4) \in \overline{F} \setminus O$ we have

\[
\{\delta((q_1, q_2, q_3, q_4), d^i) \mid \lambda \in \mathbb{N}\} = \{(q_1, q_2, q_3, x) \mid x \in \{0, 1\}\} \subseteq \overline{F}.
\]

Hence, if $u = d$ then $u$ covers all rejecting states of of $Q_\tau$.

Now suppose that $u \in \Sigma^*$ covers some rejecting state $q = (q_1, q_2, q_3, 1) \in Q_\tau$. By Item (1.), either $u \in \{d\}^*$ or $u \in \{a, b\}^* \setminus (\{a\}^* \cup \{b\}^*)$. We show that $u \in \{d\}^*$, by supposing that $\#_a(u) > 0$ and deriving a contradiction. Since $\mu$ is prime, there exists $\lambda \in \mathbb{N}$ satisfying $\lambda \cdot \#_a(u) \equiv -q_1 \mod \mu$. Therefore the first component of $\delta(q, u^\lambda)$ is $0$ and its fourth component is $1$, hence it is accepting, which contradicts the assumption that $u$ covers $q$. \(\triangleleft\)
We introduced in this work powerful techniques to treat the words in composite. However, we still have hope to find a way to adapt our techniques: maybe, instead of trying to cover rejecting states by Claim 21.3, there exist (non-permutation) DFAs that bounding the number of states is not as useful in general as it is for permutation DFAs: while it is known that every permutation DFA with a single rejecting state is prime [9], there exist (non-permutation) DFAs with a single rejecting state that are composite. Hence, by Lemma 12, there exists a set \(\{v_1, v_2, \ldots, v_m\}\) ⊆ \(Q\) which is a solution since \(|X| ≤ k\) due to \(W \cap \{d\}^* ≠ \emptyset\), and for each \(C \in \mathcal{F}\) we have \(X \cap C \neq \emptyset\).

### 6 Discussion

We introduced in this work powerful techniques to treat the DECOMP problem for permutation DFAs. We discuss how they could help solving the related questions that remain open:

- How do the insights obtained by our results translate to the general setting?
- How can we use our techniques to treat other variants of the DECOMP problem?

**Solving the general setting.** The techniques presented in this paper rely heavily on the group structure of transition monoids of permutation DFAs, thus cannot be used directly in the general setting. They still raise interesting questions: Can we also obtain an FPT algorithm with respect to the number of rejecting states in the general setting? Some known results point that bounding the number of states is not as useful in general as it is for permutation DFAs: while it is known that every permutation DFA with a single rejecting state is prime [9], there exist (non-permutation) DFAs with a single rejecting state that are composite. However, we still have hope to find a way to adapt our techniques: maybe, instead of trying to cover rejecting states, we need to cover rejecting behaviours of the transition...
monoid. Another way to improve the complexity in the general setting would be to bound the width of DFAs: we defined here a family of DFAs with polynomial width, do there exist families with exponential width? If this is not the case (i.e., every composite DFA has polynomial width), we would immediately obtain a PSPACE algorithm for the general setting.

Variants of the Decomposition problem. In this work, we focused on the Bound-Decomposition problem, that limits the number of factors in the decompositions. Numerous other restrictions can be considered. For instance, the Fragmentation problem bounds the size of the factors: Given a DFA $A$ and $k \in \mathbb{N}$, can we decompose $A$ into DFAs of size smaller than $k$? Another interesting restriction is proposed by the Compression problem, that proposes a trade-off between limiting the size and the number of the factors: given a DFA $A$, can we decompose $A$ into DFAs $(A_i)_{1 \leq i \leq k}$ satisfying $\sum_{i=1}^{k}|A_i| < |A|$? How do these problems compare to the ones we studied? We currently conjecture that the complexity of the Fragmentation problem matches the Decomposition problem, while the complexity of the Compression problem matches the Bound-Decomposition problem: for commutative permutation DFAs, the complexity seems to spike precisely when we limit the number of factors.

References

A Algorithm: Section 3

Algorithm 1 NP-algorithm for the DECOMP problem for permutation DFAs.

Function isComposite($A = (\Sigma, Q, q_I, \delta, F)$: permutation DFA)

```
foreach $p \in Q \setminus F$ do
    guess $U$ with \{ $p$ \} \subseteq U \subseteq Q \setminus F$ /* guess rejecting state $U$ of some orbit-DFA, such that $U$ contains rejecting state $p$ of $A$ */
    if not cover($A, p, U$) then return False
return True
```

Function cover($A = (\Sigma, Q, q_I, \delta, F)$: permutation DFA, $p \in Q \setminus F$, $U \subseteq Q \setminus F$)

```
$C_U^{old} = \emptyset$
$C_U := \{ U \}$
while $C_U \neq C_U^{old}$ and $|C_U| < |Q|$ do
    $C_U^{old} := C_U$
    $C_U := C_U \cup \{ \delta(S, \sigma) \mid S \in C_U, \sigma \in \Sigma \}$
if $|C_U| \geq |Q|$ then return False /* check that orbit-DFA is factor */
foreach $S \in C_U$ do
    if $q_I \in S$ then return True /* check that $U$ is reachable from the initial state of the orbit-DFA */
return False
B  Algorithm: Section 4

Algorithm 2  Deterministic and non-deterministic version of the algorithm solving the
DECOMP problem for commutative permutation DFAs.

Function \texttt{isComposite}(A = (\Sigma, Q, q_I, \delta, F)) : commutative permutation DFA
\begin{algorithmic}
\For {\( p \in Q \setminus F \)}
\State \texttt{cover\_found} := False
\For {\( q \in Q \setminus F \) with \( q \neq p \)}
\If {\texttt{cover}(A, p, q)}
\State \texttt{cover\_found} := True /* covering \( p \) with \( w_{p,q} \) */
\EndIf
\EndFor
\EndFor
\If {\texttt{not} \texttt{cover\_found}}
\Return False /* no cover found for \( p \) */
\EndIf
\Return True /* all state \( p \) are covered */
\end{algorithmic}

Function \texttt{cover}(A = (\Sigma, Q, q_I, \delta, F)) : commutative permutation DFA, \( p, q \in Q \setminus F \)
\begin{algorithmic}
\State \( s := q \)
\While {\( s \neq p \)} /* eventually \( s = p \) \( A \) is a permutation DFA */
\State \( s := \texttt{mimic}(p, q, s) \) /* thus \( s := \delta(s, w_{p,q}) \) */
\If {\( s \in F \)}
\Return False /* contradiction of covering */
\EndIf
\EndWhile
\Return True /* encountered \( p \) again without hitting state in \( F \) */
\end{algorithmic}

Function \texttt{mimic}(A = (\Sigma, Q, q_I, \delta, F)) : commutative permutation DFA, \( p, q, s \in Q \setminus F \)
\begin{algorithmic}
\Assumption \(|\Sigma|\) is fixed, let \( \Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \)
\For {\( 1 \leq x_1 + \cdots + x_i \leq |Q| \) do} /* possible since \(|\Sigma|\) is fixed */
\If {\( \delta(p, \sigma_1^{x_1} \sigma_2^{x_2} \cdots \sigma_m^{x_m}) = q \)}
\Return \( \delta(s, \sigma_1^{x_1} \sigma_2^{x_2} \cdots \sigma_m^{x_m}) \)
\EndIf
\EndFor
\end{algorithmic}

Function \texttt{mimic}(A = (\Sigma, Q, q_I, \delta, F)) : commutative permutation DFA, \( p, q, s \in Q \setminus F \)
\begin{algorithmic}
\Assumption this algorithm is allowed to use non-determinism
\State \( p' := p, \ell := 0 \)
\While {\( p' \neq q \) and \( \ell < |Q| \)}
\State \( \sigma \in \Sigma \) /* iteratively construct \( w_{p,q} \) of length \( \ell \) */
\State \( p' := \delta(p', \sigma), s := \delta(s, \sigma), \ell := \ell + 1 \)
\EndWhile
\If {\( \ell = |Q| \)}
\Return \texttt{error} else \Return \( s \) /* check \( q = \delta(p, w_{p,q}) \) */
\EndIf
\end{algorithmic}
C Algorithms: Section 5

Algorithm 3 LOGSPACE-algorithm solving the \textsc{Bound-Decomp} problem for unary DFAs.

\textbf{Function} \text{isBoundedComposite}(A = \langle \{a\}, Q, q_I, \delta, F \rangle: \text{unary DFA}, \text{integer } k)

\begin{algorithmic}
\If {A is permutation DFA}
\ForEach {binaryString wordCombination $\in \{0, 1\}^{\log |Q|}$ with $\leq k$ ones}
\Comment{wordCombination represents current set in $2^W$}
\If {testWordCombination(A, wordCombination)} \Return True \EndIf
\EndIf
\Else
\Call {7, Algorithm 1}
\EndIf
\EndAlgorithmic

\textbf{Function} \text{testWordCombination}(A = \langle \{a\}, Q, q_I, \delta, F \rangle: \text{unary DFA},
\text{wordCombination: binaryString})

\begin{algorithmic}
\ForEach {$q \in Q \setminus F$}
\If {not \text{cover}(A, q, \text{wordCombination})} \Return False \EndIf
\EndFor
\Return True
\EndAlgorithmic

\textbf{Function} \text{coverBySet}(A = \langle \{a\}, Q, q_I, \delta, F \rangle: \text{unary DFA}, q \in Q \setminus F,
\text{wordCombination: binaryString})

\begin{algorithmic}
\ForEach {int $i$ with wordCombination$[i] \neq 1$}
\Comment{Go through all $\leq k$ words in the set and test if $q$ is covered}
\Compute $p_1 := i$th prime divisor of $|Q|$
\If {\text{cover}(A, q, \delta(q, a^{i|Q|/p_1}))} \Return True \EndIf
\EndFor
\Return False
\EndAlgorithmic

Algorithm 4 NP-algorithm solving the \textsc{Bound-Decomp} problem for commutative permutation DFAs.

\textbf{Function} \text{isBoundedComposite}(\text{commutative permutation DFA } A, \text{integer } k)

\begin{algorithmic}
\Comment{guess $W := \{w_i \in \Sigma^{\leq |Q|} \mid i \leq k\}$}
\ForEach {$p \in Q \setminus F$}
\If {not \text{cover}(A, p, W)} \Return False \EndIf
\EndFor
\Return True
\EndAlgorithmic

\textbf{Function} \text{cover}(\text{commutative permutation DFA } A, \text{state } p, \text{set of words } W)

\begin{algorithmic}
\ForEach {$w_i \in W$}
\Compute $Q_{q,w_i} := \{\delta(q, w_i^\lambda) \mid \lambda \leq |Q|\}$
\If {$Q_{q,w_i} \cap F = \emptyset$} \Return True \EndIf
\EndFor
\Return False
\EndAlgorithmic